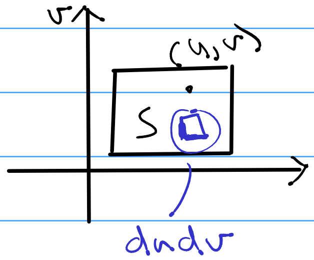
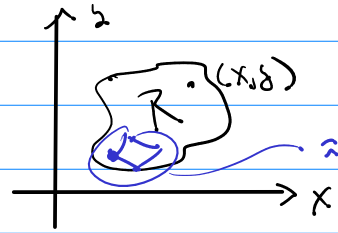
Math 344

2D



$x = x(u,v)$
$y = y(u,v)$ $\longrightarrow$



$\approx |r_u \times r_v| \, du\,dv$

let

$\vec{r} = \langle x(u,v), y(u,v) \rangle$

$$\underbrace{\iint_{S} f(x(u,v), y(u,v))\, \boxed{|r_u \times r_v|}\, du\,dv}_{u,v\text{'s}} = \underbrace{\boxed{\iint_{R} f(x,y)\, dx\,dy}}_{x,y\text{'s}}$$

let $x = x(u,v)$ or $\vec{r} = \langle x(u,v), y(u,v) \rangle$
$y = y(u,v)$

So $$\iint_R f(x,y)\,dx\,dy = \underbrace{\iint_{S}}_{\text{in } u,v\text{'s}} f(x(u,v), y(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\,dv$$

Need $|\vec{r}_u \times \vec{r}_v|$ let ① $\vec{r}_u = \langle x_u, y_u, 0 \rangle$
$\vec{r}_v = \langle x_v, y_v, 0 \rangle$

② $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \vec{k} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \cancel{-0|\;|+0|\;|}$$

$$= \langle 0, 0, x_u y_v - x_v y_u \rangle$$

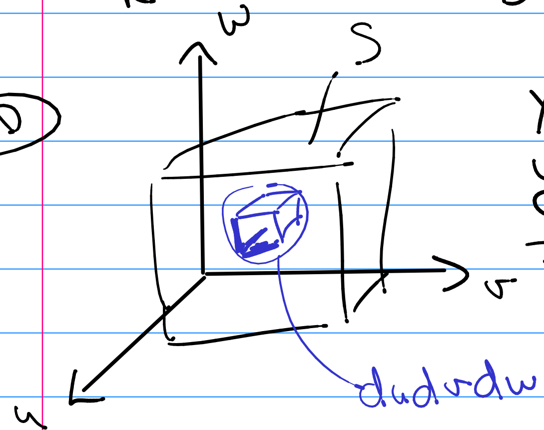
③ $$|\vec{r}_u \times \vec{r}_v| = x_u y_v - x_v y_u \underset{\text{by observation}}{=} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = ?$$

$x(u,v) \to x_u = ?$ $x_v ?$
$y(u,v) \to y_u = ?$ $y_v ?$

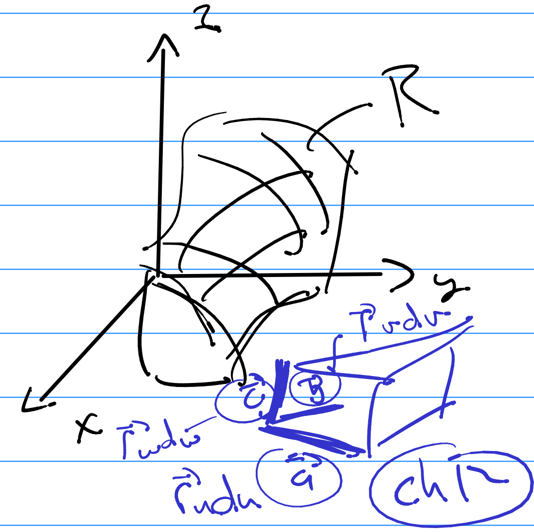
$x = x(u,v) \quad y = y(u,v)$

2D $$\iint_R f\,dxdy = \iint_S f(x(u,v), y(u,v)) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} dxdy$$

3D



$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$



$$\iiint_R f\,dxdxdz = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \, \square \, dudvdw$$

parrallelpiped

$|\vec a \cdot (\vec b \times \vec c)|$

$$\left|\begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix}\right|$$

use a "new" notation

2D $$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

3D $$\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|$$

Jacobian

let $x = x(u)$

1D $$\int_R f(x)\,dx = \int_S f(x(u))\,x'(u)\,du$$

2D polar transform! $x = r\cos\theta$

$y = r\sin\theta$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

in r,θ's

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

3D

$$\iiint_R f\,dx\,dy\,dz = ?$$

use a spherical trans.

$$\begin{bmatrix} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{bmatrix}$$

$$= \iiint_S f \,\boxed{\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|}\, d\rho\, d\theta\, d\phi$$

ρ,θ,ϕ

$$\begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi\begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - 0 + (-\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \text{etc.}\ldots = \rho^2\sin\phi$$

ex $$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f \boxed{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}\,du\,dv\,dw$$

where:

$$T: \begin{cases} x = u + vw \\ y = v + uw \\ z = w + uv \end{cases}$$

Jacobian:

$$\begin{vmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{vmatrix} = \cdots =$$