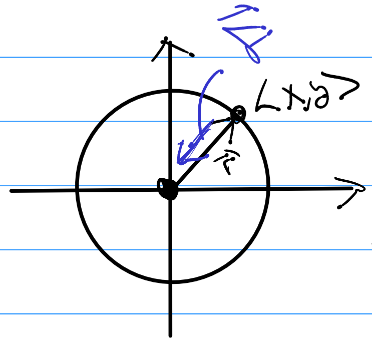
Math 394

Vector Fields: $\mathbf{F}(x,y,z) = \langle \underbrace{f_1, f_2, f_3}_{\text{Scalar functions}} \rangle$ $\quad \vec{F}(x,y) = \langle \underbrace{f_1, f_2}_{\text{Scalar functions}} \rangle$

ex



$|\text{force of gravity}| \propto \frac{Mm}{|\vec{r}|^2}$

$|\text{Force of gravity}| = \frac{mMG}{|\vec{r}|^2}$

direction is towards center of mass. $-\vec{r}$

→ unit vector $\frac{-\vec{r}}{|\vec{r}|}$

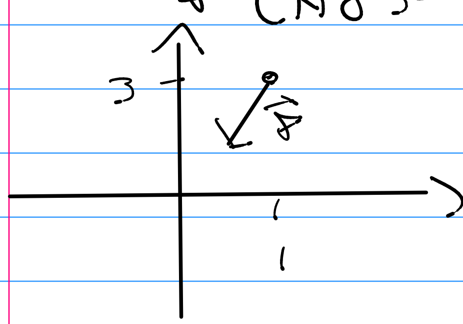
$$\mathbf{F} = \underbrace{\left(\frac{mMG}{|\vec{r}|^2}\right)}_{\text{mag}} \circ \underbrace{\left(\frac{-\vec{r}}{|\vec{r}|}\right)}_{\text{unit vector}} = \frac{-mMG}{|\vec{r}|^3}\vec{r}$$

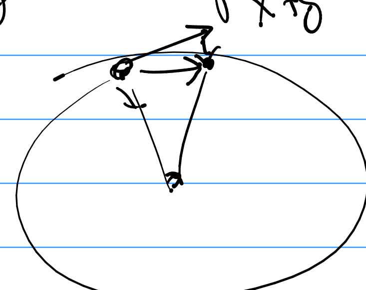
b/c $\vec{r} = \langle x,y \rangle$ $\quad |\vec{r}| = \sqrt{x^2+y^2}$

So the gravitation (vector) field

$$\mathbf{F} = \frac{-mMG}{(x^2+y^2)^{3/2}} \langle x, y \rangle \quad \text{2D}$$

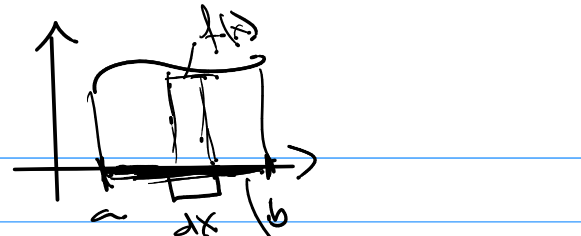
$$\vec{F}(x,y) = \left\langle \frac{-mMG\,x}{\sqrt{x^2+y^2}}, \frac{-mMG\,y}{\sqrt{x^2+y^2}} \right\rangle$$



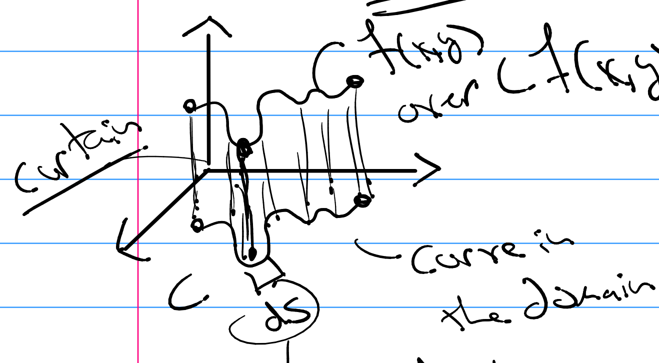


16.2

Calc 1 $\int_a^b f(x)\,dx$



add $f(x)dx$ over line from $x=a$ to $x=b$



over C $f(x,y)$

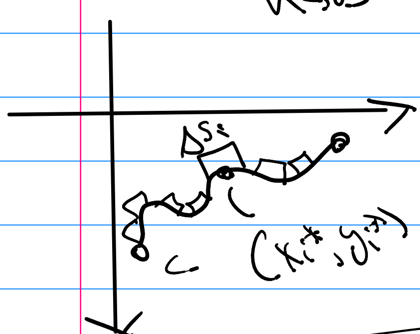
$f: \mathbb{R}^2 \to \mathbb{R}$

domain

add all $f(x,y) \cdot ds$ over curve C

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i = \int_C f(x,y)\,ds}$$

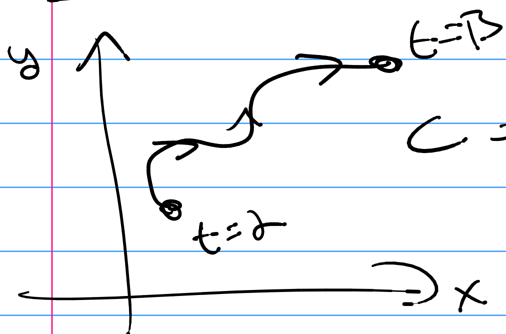
Line Integral of f along curve C



$$\boxed{\int_C f(x,y)\,ds = ?}$$

Step 1: $C = ?$

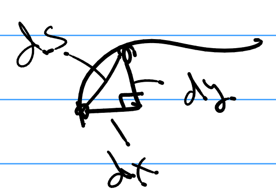
represented in parametric form.



$C = \langle x(t), y(t) \rangle$

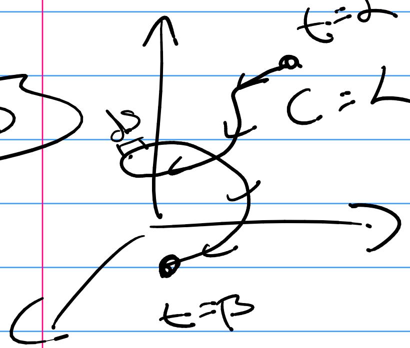
$f(x,y) = f(x(t), y(t))$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$



So. $$\int_C f(x,y)\,ds = \int_{t=\alpha}^{t=\beta} f(x(t),y(t))\sqrt{(\underbrace{x'}_{\frac{dx}{dt}})^2+(\underbrace{y'}_{\frac{dy}{dt}})^2}\,dt$$

3D



$C = \langle x(t), y(t), z(t)\rangle$ space curve

$$\int_C f(x,y,z)\,ds$$

$$\int_C f(x,y,z)\,ds = \int_{t=\alpha}^{t=\beta} f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

We can use one notation for 2D or 3D

$\vec{r} = \langle x(t), y(t)\rangle$ $\qquad$ $\vec{r} = \langle x(t), y(t), z(t)\rangle$

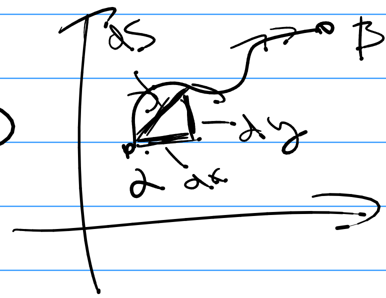
$\vec{r}\,' = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$ $\qquad$ $\vec{r}\,' = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$

$|\vec{r}\,'| = \left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2\right)^{1/2}$ $\qquad$ $|\vec{r}\,'| = \left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2\right)^{1/2}$

$$\boxed{\int_C f(x,y)\,ds = \int_\alpha^\beta f(\vec{r})\,|\vec{r}\,'|\,dt}$$

Line integral with respect to arc length ds

① $\int_C f(\vec{r})\,ds = \int_\alpha^\beta f(\vec{r})\,|\vec{r}\,'|\,dt$

② 

adding $f\,dx$ $\qquad C = \langle x(t), y(t)\rangle$

adding $f\,dy$ $\qquad x = x(t) \to dx = x'(t)\,dt$

$y = y(t) \to dy = y'(t)\,dt$

line integral with respect to x $\quad \int_C f\,\boxed{dx} = \int_\alpha^\beta f(x(t), y(t))\,x'(t)\,dt$

line integral with respect to y $\quad \int_C f\,dy = \int_\alpha^\beta f(x(t), y(t))\,y'(t)\,dt$

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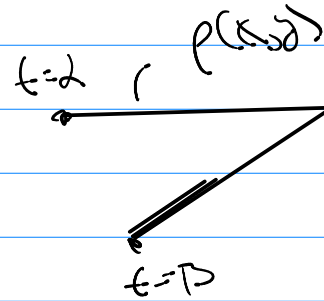
ex5 **Center of mass**

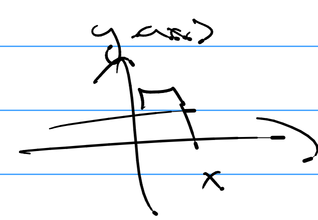
$\bar{X} = \frac{1}{\text{total mass}} \cdot$ all moments (mass · x) w.r.t. x-coord



$C: \langle x(t), y(t)\rangle$

total mass $= \int_C \rho\,ds$

Moments with respect to x-coord $= \boxed{\int_C x\,\rho\,ds}$



$$\int_C x \rho \, ds$$

Given $C = \langle x(t), y(t) \rangle \quad \alpha \leq t \leq \beta$

$\rho(x,y)$

$$= \int_{\alpha}^{\beta} x(t)\, \rho\big(x(t), y(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
