Math 399

16.5 Curl, Divergence, Laplace Operator..

Notation: b/c ∇f = gradient of a scalar function/field

$$\nabla f = \langle f_x, f_y, f_z \rangle$$ vector output

so ∇f is a vector field

Notice: $\boxed{\nabla f} = \langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \rangle$

call $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

① $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Note: $\vec{F}$ is 2D let $\vec{F} = \langle P, Q, 0 \rangle$

② $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$

terms a) $\text{curl}(\vec{F}) = \vec{0}$ we call $\vec{F}$ irrotational

b) $\text{div}(\vec{F}) = 0$ we call $\vec{F}$ incompressible

---

ex $\vec{F}(x,y,z) = \langle \frac{\sqrt{x}}{1+z}, \frac{\sqrt{y}}{1+x}, \frac{\sqrt{z}}{1+y} \rangle$

$\text{curl}(\vec{F})$? $\text{div}(\vec{F})$?

a) $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\sqrt{x}}{1+z} & \frac{\sqrt{y}}{1+x} & \frac{\sqrt{z}}{1+y} \end{vmatrix}$

$$= \left\langle \frac{-\sqrt{z}}{(1+y)^2} - 0 , \frac{-\sqrt{x}}{(1+z)^2} - 0 , \frac{-\sqrt{y}}{(1+x)^2} - 0 \right\rangle$$

$$= -\left\langle \frac{\sqrt{z}}{(1+y)^2} , \frac{\sqrt{x}}{(1+z)^2} , \frac{\sqrt{y}}{(1+x)^2} \right\rangle$$

b) $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\sqrt{x}}{1+z}, \frac{\sqrt{y}}{1+x}, \frac{\sqrt{z}}{1+y} \right\rangle$

$$= \frac{1}{2\sqrt{x}(1+z)} + \frac{1}{2\sqrt{y}(1+x)} + \frac{1}{2\sqrt{z}(1+y)}$$

Ex $\quad \vec{F} = \langle e^x \sin y , e^y \sin z , e^z \sin x \rangle$

$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix}$

$$= \langle 0 - e^y \cos z , 0 - e^z \cos x , 0 - e^x \cos y \rangle$$

$$= -\langle e^y \cos z , e^z \cos x , e^x \cos y \rangle$$

$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$

$$= e^x \sin y + e^y \sin z + e^z \sin x$$

Note: So for this problem $\text{Div}(\vec{F})$ = sum of it's own components

Applications

① Conservative Vector Fields .. $\vec{F} = \nabla f$

Th^n if $\vec{F}$ is conservative then $\text{curl}(\vec{F}) = \vec{0}$

Th^n if ① $\vec{F} = \langle P, Q, R \rangle$ is defined for all $\mathbb{R}^3$
② P, Q, R have cont. partials
③ and $\text{curl}(\vec{F}) = \vec{0}$

then: $\vec{F}$ is conservative and $\vec{F} = \nabla f$

ex $\vec{F} = \langle y^3 z^3, 2xyz^3, 3xy^2z^2 \rangle$

① defined for all x, y, z $\boxed{OK}$
② cont partials $\boxed{OK}$

③ $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$

$$= \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$ $\boxed{\text{Yep!}}$

So $\langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle = \nabla f$ is true

Find f?

(*) $\langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle = \langle f_x, f_y, f_z \rangle$

Step 1 $\quad y^2 z^3 = \frac{\partial}{\partial x} f$

const according to x

$$\boxed{f(x,y,z) = xy^2z^3 + C(y,z)}$$

Step 2 $\quad f_y = 2xyz^3 + \underline{\frac{\partial}{\partial y} C(y,z)}$

Compare.

from above ✱ $\quad f_y = 2xyz^3$

So $\frac{\partial}{\partial y} C(y,z) = 0$

so $C(y,z)$ has no y's

$\therefore \; C(y,z) = C(z)$

So $\boxed{f(x,y,z) = xy^2z^3 + C(z)}$

Step 3 $\quad f_z = 3xy^2z^2 + \frac{\partial}{\partial z} C(z)$

Compare

but by ✱ $\quad f_z = 3xy^2z^2$

So $\frac{\partial}{\partial z} C(z) = 0$

no z's in $C(z)$

ans $\quad \boxed{f(x,y,z) = xy^2z^3 + C}$

Some scalar.