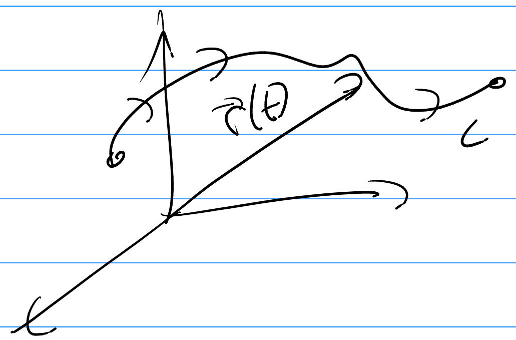
Math 344

Ch 16 — 11 probs @ 10 pts — 100 pts = 100%

exam 4

Vector Calculus — function: $\mathbb{R}^{1 \text{ or } 2} \to$ Vector

16.1



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

→ Vector Fields

$\vec{F} = \langle P, Q \rangle$ 2D $P(x,y), Q(x,y)$

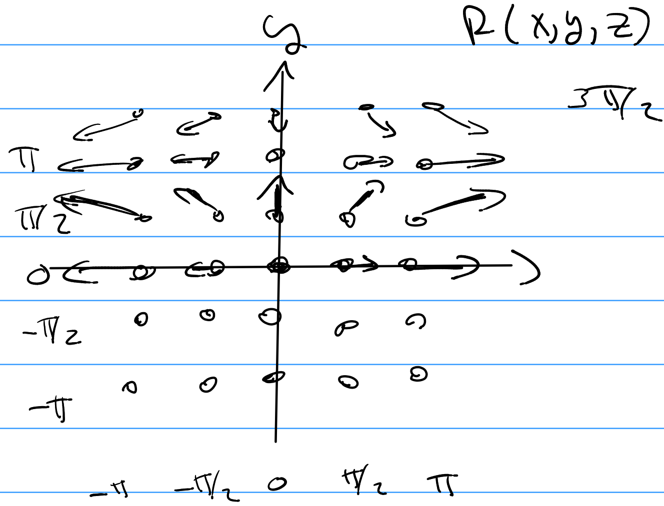
$\vec{F} = \langle P, Q, R \rangle$ 3D $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$

16.1 $\vec{F} = \langle P, Q \rangle$

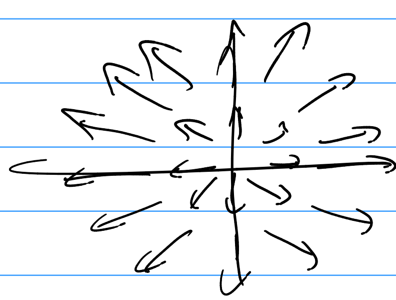
draw a vector field

ex

$\vec{F} = \langle x, \sin(y) \rangle$



ex



$\langle x, y \rangle$

$\boxed{16.2}$

I will give you $\boxed{\int_C f\,ds = \int_\alpha^\beta f(\vec{r})\,|\vec{r}'|\,dt}$

① $\int_C f\,ds \overset{?}{=}$

(ex) $f(x,y) = x^2+y^2$ $\quad$ $\vec{r}(t)$ is C

$\vec{r}(t) = \langle \sin(2t), \cos(2t) \rangle$

$t \in [0,1]$

$$\int_C (x^2+y^2)\,ds = \int_0^1 \left[\sin^2(2t) + \cos^2(2t)\right]\sqrt{4\cos^2(2t) + 4\sin^2(2t)}\,dt$$

$$= \int_0^1 (1)(2)\,dt = \int_0^1 2\,dt = \boxed{2}$$

② given $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_\alpha^\beta \left(\vec{F}(\vec{r})\cdot \vec{r}'\right)dt = \int_C P\,dx + Q\,dy + R\,dz}$

$F = \langle P, Q, R \rangle$

(ex) $\vec{F} = \langle x+y^2, xz, y+z \rangle$

C is $\vec{r}(t) = \langle t^2, t^3, -2t \rangle \quad t \in [0,1]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\langle t^2+t^6, -2t^3, t^3-2t \rangle \cdot \langle 2t, 3t^2, -2 \rangle\right)dt$$

$$= \boxed{\int_0^1 (2t^3+2t^7) + (-6t^5) + (-2t^3+4t)\,dt}$$

= finish ...

or $\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$

Same ans.

$\boxed{16.3}$ $\int_C \vec{F} \cdot dr$ but $\vec{F} = \nabla f$

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

① is $\vec{F} = \nabla f$?

all points here.

is $\vec{F}$ conservative? (Why?)

ex $F = \langle P, Q \rangle = \langle \underbrace{\sin(xy)}_{P}, \underbrace{\sin(xy)}_{Q} \rangle$

check $Q_x = P_y \longrightarrow$

$Q_x = y\cos(xy)$

$P_y = x\cos(xy)$

$\neq$ so $\boxed{\text{No}}$

② use $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

ex $\vec{F} = \langle \underbrace{y\cos x}_{P}, \underbrace{\sin x}_{Q} \rangle$

C is $\vec{r}(t) = \langle t^3 + t, 2t - t^2 \rangle$ $t = [0, 1]$

a) check: $\vec{F} = \nabla f$ $\quad Q_x = \cos x \quad P_y = \cos x$

yep! so f?

$\langle y\cos x, \sin x \rangle = \langle f_x, f_y \rangle$

$f_x = y\cos x \rightarrow \boxed{f = y\sin(x) + c(y)}$

Use and $f_y = \sin x + c'(y)$ ∴ $c'(y) = 0$

Know from $\vec{F}$: $f_y = \sin(x)$ $c(y) = C$

∴ $\boxed{f = y\sin(x) + C}$

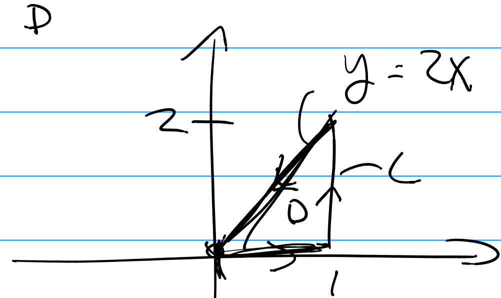
@ $t=0$
$\vec{r}(0) = \langle 0,0 \rangle$
@ $t=1$
$\vec{r}(1) = \langle 3,1 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \sin(2) - 0$$
$$= \boxed{\sin(2)}$$

16.4 $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$ $\vec{F} = \langle P, Q \rangle$

① $\oint_C \vec{F} \cdot d\vec{r}$ use $\iint_D (Q_x - P_y)\, dA$

$\vec{F} = \langle xy, x^2y^3 \rangle$



$$\iint_D (2xy^3 - x)\, dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\, dy$$

② $\iint_D (1)\, dA = \oint_C \vec{F} \cdot d\vec{r}$

16.5 ① $\nabla \times \vec{F}$ ② $\nabla \cdot \vec{F}$

16.6 ① $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$u, v \in D$

Setup

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

ex $\vec{r} = \langle u, vu, u^2+v^2 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & v & 2u \\ 0 & u & 2v \end{vmatrix} = \langle \ , \ , \ \rangle$$

16.7 $$\iint_S f \, dS = \iint_D f(\vec{r}) \, |\vec{r}_u \times \vec{r}_v| \, dA$$

Setup ex

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n}) \, dS = \iint_D \vec{F}(\vec{r}) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$