Math 321

Q. prove $\sqrt{2}$ is irrational with lemma.

pf. lemma: a^2 even $\to$ a even.

pf. (by direct proof of contrapositive)

$(a^2 \text{ even} \to a \text{ even}) \equiv (\underline{a \text{ odd}} \to \underline{a^2 \text{ odd}})$

assume a is odd. This is $a = 2K+1$, K an int.

so $(a)^2 = (2K+1)^2$

$a^2 = 4K^2 + 4K + 1 = 2(2K^2 + 2K) + 1$

$a^2 = 2(\text{integer}) + 1$ so $\underline{a^2 \text{ is odd}}$

$\blacksquare$

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back to $\sqrt{2}$ is irrational. Try contradiction.

Assume $\neg(\sqrt{2}$ is irrational$)$ is $\sqrt{2}$ is rational

So $\boxed{\sqrt{2} = \frac{a}{b}}$ such that ① a, b are int

② $b \neq 0$

③ no common factors

now $(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$. gives $2 = \frac{a^2}{b^2}$. gives $2b^2 = a^2$

now we see a^2 is even $\to$ (by lemma) a is even.

let $a = 2K$, K an integer

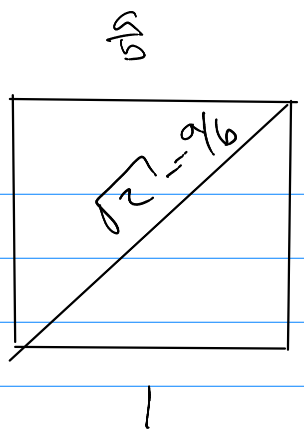
gives $2b^2 = (2K)^2$. gives $2b^2 = 4K^2$

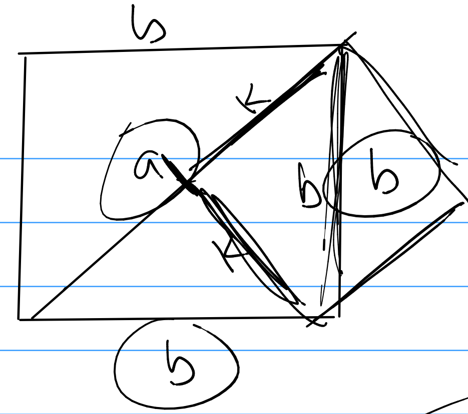
so $b^2 = 2K^2$

b^2 is even $\to$ (by lemma) b is even.

Now: a is even and b is even and they have no common factors. $\equiv F$

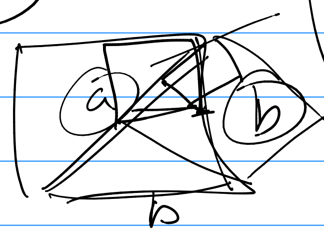
$(\sqrt{2}$ is rational$) \equiv F \to (\sqrt{2}$ irrational$) \equiv T$. $\blacksquare$



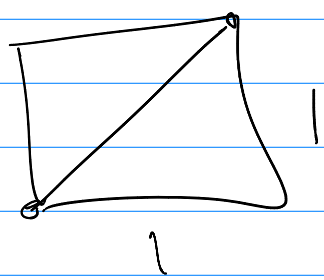


$\boxed{\begin{array}{l} 2b^2 = a^2 \\ a^2 \text{ even} \\ \to a \text{ even} \end{array}}$

$\sqrt{2} = \frac{a}{b}$



$\frac{a}{b} = a\left(\frac{1}{b}\right)$ 1, 2, 3, 4, ...



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techniques

① forward / backward reasoning.

$\boxed{\text{Start}}$ ← where am I?

forward reasoning

backward reasoning

$\boxed{\text{end}}$ ← what is my goal?

② Experiment

③ Modify proofs you know.

ex $\sqrt{3}$ is irrational.

pf $\neg(\sqrt{3} \text{ is irrational}) = \sqrt{3}$ is rational

$\sqrt{3} = \frac{a}{b}$

① a, b are ints

② $b \neq 0$

③ no common factors

$\to 3 = \frac{a^2}{b^2} \to 3b^2 = a^2$

$2b^2 = a^2$

lemma: a^2 has a factor of $3 \Rightarrow a$ has a factor of 3

pf?: know factor of $3 \to 3 \cdot k$

not factor of $3 \to 3k+1$, $3k-1$

lemma: a^2 has a factor of 2 $\to a$ has a factor of 2

$3b^2 = a^2$, a^2 has a factor of $3 \to$ a has a factor of 3.

$\to a = 3 \cdot k$, k is an int

So $3b^2 = (3k)^2 \to 3b^2 = 9k^2$

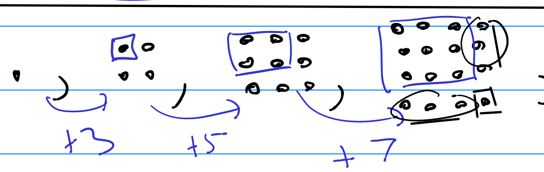
$\to b^2 = 3k^2$ so b^2 has a factor of 3

$\to$ b has a factor of 3.

Q prove every odd is a diff. of two squares.

$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$

+1, +3, +5, +7



$16 - 9 = 7$

$(4)^2 - (3)^2 = 7$

$9 - 4 = (3)^2 - (2)^2 = 5$

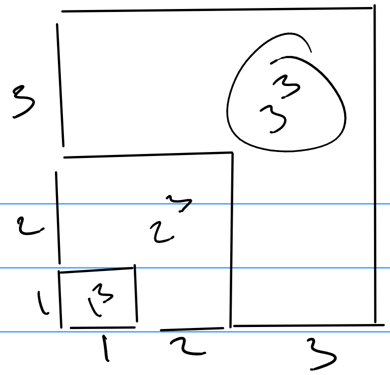
let smaller square be $(k)^2$

next square is $(k+1)^2$

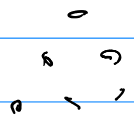
Diff: $(k+1)^2 - k^2$

$= k^2 + 2k + 1 - k^2$

$= 2k+1$



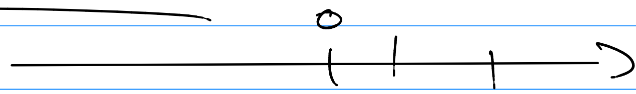
Sum of first 3 cubic numbers is the square of the third triangular number.



$$1^3+2^3+3^3 = (1+2+3)^2 = \left(\frac{3(4)}{2}\right)^2$$

1.8 #7 $|x+y| \le |x|+|y|$

$$|\square| = \begin{cases} \square & \text{if } \square \ge 0 \\ -\square & \text{if } \square < 0 \end{cases}$$

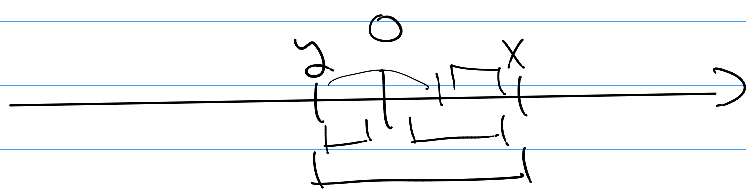


Case 1 x, y are not neg

Case 2 x, y are neg → $|x| = -x$, $|y| = -y$ $|x+y| = -(x+y)$

Case 3 x is not neg, y is neg

~~Case 4 x is neg, y is not neg~~



$|x|+|y|$

$|x+y|$

$\exists x P(x)$ non-constructive. Show a witness exists.

(ex) ⓐ fact: all numbers are → ① rational (or) → ② irrational

ⓒ $\sqrt{2}$ is irrational

Statement Some (irrational) irrational is a rational number.

Pf consider $(\sqrt{2})^{\sqrt{2}}$. It's a real number so --

① $(\sqrt{2})^{\sqrt{2}}$ could be rational, then $\sqrt{2}^{\sqrt{2}}$ is the witness.

② $(\sqrt{2})^{\sqrt{2}}$ could be irrational

try $\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{2} = \frac{2}{1}$ is rational.

so if $\sqrt{2}^{\sqrt{2}}$ is irration then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is the witness.

Exam 1 12 probs @ 10pts each

110pts = 100%

(1.1/1.2) (3 probs) ① give the truth table everyone should know.

② eng ⇄ syn

③ truth table (two vars)

(1.3) (2 probs)

① Show $\square \equiv \triangle$

② use logical equiv. to simplify expression

1.4/1.5 ① eng $\rightleftarrows$ sym

1.6 (2 probs)

① given premises → you state conclusions.

② given wrong arguments. what are fallacies?

1.7/1.8 (4 probs)

① direct (algebra based)

② $\sqrt{2}$ is irrational (with lemma)

③ cases after contraposition

④ non-constructive $\exists x\, P(x)$
