Math 321

Q's

2.4 Sums: $\{a_n\}$ $n=0,1,2,\dots \xrightarrow{\text{seq}} a_0, a_1, a_2, \dots$

Sum $\displaystyle\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n$

ex Seq: $\{2^n\}$ $n = 0,1,2,\dots,27$

$1, 2, 4, 8, 16, \dots, 2^{27}$

Sum: $\displaystyle\sum_{k=0}^{27} 2^k = 1+2+4+8+\dots+2^{27} = ?$

Sums to know: $\displaystyle\sum_{k=1}^{n} 1 = \underbrace{1+1+1+\dots+1}_{n} = n$

$\displaystyle\sum_{k=1}^{n} k = 1+2+3+4+\dots+n$

$\quad\quad\; n + (n-1) + (n-2) + \dots + 1$

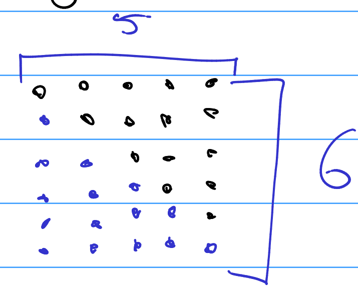
$n(n+1) = (n+1)+(n+1)+(n+1)+\dots+(n+1)$

$1+2+3+4+5 = \frac{5\cdot 6}{2}$

$5+4+3+2+1$

$6+6+6+6+6 = 5\cdot 6$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$



$a_{k+1} - a_{k-1}$

telescoping Sum

$\displaystyle\sum_{k=1}^{n} a_{k+1} - a_k$

$= \underbrace{(a_2 - a_1)}_{k=1} + \underbrace{(\cancel{a_3} - \cancel{a_2})}_{k=2} + \underbrace{(\cancel{a_4} - \cancel{a_3})}_{k=3} + \dots + \underbrace{(a_{n+1} - \cancel{a_n})}_{k=n}$

$= -a_1 + a_{n+1}$

Consider: $(k+1)^3 - (k)^3 = k^3 + 3k^2 + 3k + 1 - k^3$

$\boxed{(k+1)^3 - (k)^3} = \boxed{3k^2 + 3k + 1}$

$\displaystyle\sum_{k=1}^{n} (k+1)^3 - (k)^3 = \sum_{k=1}^{n} 3k^2 + 3k + 1$

$$\boxed{\sum_{k=1}^{n} \left((k+1)^3-(k)^3\right)} = 3\boxed{\sum_{k=1}^{n} k^2} + 3\boxed{\sum_{k=1}^{n} k} + \boxed{\sum_{k=1}^{n} 1}$$

$$3\sum_{k=1}^{n} k = 3\,\frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} 1 = n$$

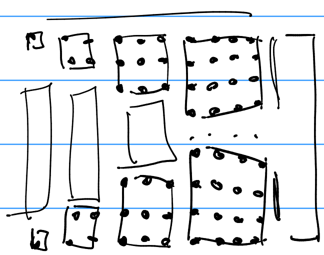
$$(\cancel{2^3}-1^3)+(\cancel{3^3}-\cancel{2^3})+(\cancel{4^3}-\cancel{3^3})+\cdots+(n+1)^3-\cancel{(n)^3}$$

$$(n+1)^3-1$$

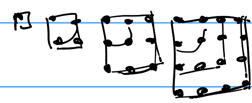
$$(n+1)^3-1 = 3\boxed{\sum_{k=1}^{n} k^2} + 3\left(\frac{n(n+1)}{2}\right) + n$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[(n+1)^3-1-n-3\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{1}{3}\,\frac{n(n+1)}{2}(2n+1) = \boxed{\frac{n(n+1)(2n+1)}{6}}$$



$2n+1$



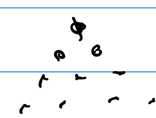
Use $(k+1)^4-(k)^4 = \cancel{k^4}+4k^3+6k^2+4k+1-\cancel{k^4}$

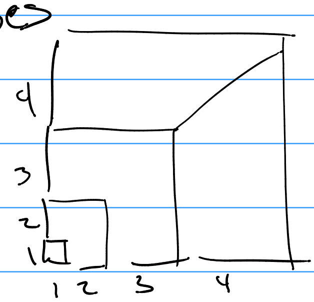
$$= 4k^3+6k^2+4k+1$$

Show: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\underbrace{1^3+2^3+3^3+4^3}_{\text{Sum of 1st 4 cubes}} = \left(\frac{4(4+1)}{2}\right)^2$$

4th triangular number





$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \dots + ar^n = \begin{cases} a\left(\frac{r^{n+1}-1}{r-1}\right) & r \neq 1 \\ a(n+1) & r = 1 \end{cases}$$

$$\boxed{S} = a + ar + ar^2 + \dots + ar^n$$
$$rS = ar + ar^2 + ar^3 + \dots + ar^{n+1}$$

$$rS - S = ar^{n+1} - a$$

$$S(r-1) = a(r^{n+1} - 1)$$
$$S = a\left(\frac{r^{n+1}-1}{r-1}\right) \quad r \neq 1$$

---

use this idea on $0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$

$$S = 0.999\ldots$$
$$10S = 9.999\ldots$$

$$10S - S = 9$$
$$9S = 9$$
$$S = 1$$

$$1 = 0.999\ldots$$

So $123.239999\ldots = 123.24$

$\boxed{2.5}$ Cardinality (sizes of infinity)

$|S|$ is the cardinality of S

if S has a fixed amount of uniq. elements $|S| = n \in \mathbb{N}$

$\to$ call S finite

$|S|$ is not finite $\to$ S is infinite.

$|\mathbb{Z}^+|$ infinite, $|\mathbb{Q}|$ infinite, $|\mathbb{R}|$ infinite

$|\text{odds}|$ infinite...

Def: $|A| = |B|$

$$f \rightarrow \begin{array}{cccc} a_1 & a_2 & a_3 & \cdots \\ | & | & | & \\ b_1 & b_2 & b_3 & \cdots \end{array}$$

f is a bijection $f: A \to B$

Def: $|\mathbb{Z}^+| = \aleph_0$

Def: if $|S|$ is finite or $|S| = \aleph_0$
call S countable

Hilbert's Grand Hotel

Hotel: $r_1, r_2, r_3, r_4, r_5, r_6, \ldots$

$$\begin{array}{ccccc} p_{11} & p_{12} & p_{13} & p_{14} & \cdots \\ p_{21} & p_{22} & p_{23} & p_{24} & \cdots \\ p_{31} & p_{32} & p_{33} & p_{34} & \cdots \\ p_{41} & p_{42} & p_{43} & p_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

0 $d=1$ $d=2$ $d=3$ $d=4$

$$\begin{array}{ccccc} \pm\frac{1}{1} & \pm\frac{1}{2} & \pm\frac{1}{3} & \pm\frac{1}{4} & \cdots \\ \pm\frac{2}{1} & \pm\frac{2}{2} & \pm\frac{2}{3} & \pm\frac{2}{4} & \cdots \\ \pm\frac{3}{1} & \pm\frac{3}{2} & \pm\frac{3}{3} & \pm\frac{3}{4} & \cdots \\ \pm\frac{4}{1} & \pm\frac{4}{2} & \pm\frac{4}{3} & \pm\frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$\frac{101}{17}$ $0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}$
skip $\pm\frac{2}{2} \notin \mathbb{Q}$

shows $|\mathbb{Q}| = \aleph_0$ (See video, textbook)

$|\mathbb{R}| = ?$

Fact: $\mathbb{R}$ is uncountable (if $[0,1] \to \mathbb{R}$ is uncountable $\mathbb{R}$ is uncountable)

Pf: use Contradiction

assume reals between 0 and 1 are countable.

So

$1 \to r_1 = 0.d_{11}d_{12}d_{13}\ldots$

$2 \to r_2 = 0.d_{21}d_{22}d_{23}\ldots$

$3 \to r_3 = 0.d_{31}d_{32}d_{33}\ldots$

$4 \to r_4 = 0.d_{41}d_{42}d_{43}\ldots$

$\vdots$

① d_{ij} are just decimals

② exclude all $\bar{9}$ decimal reals

ex $123.23999\ldots = 123.24$

So each decimal expansion is uniq.

and all reals between 0 and 1 are in this list.

Consider this number $r^* = 0.d_1 d_2 d_3 d_4 \ldots$ is between 0 and 1

$\boxed{\text{pick } d_i}$

① $d_1 \neq d_{11},\ d_1 \neq 9,\ d_1 \neq 0 \to r_1 \neq r^*$

② $d_2 \neq d_{22},\ d_2 \neq 9,\ d_2 \neq 0 \to r_2 \neq r^*$

$\vdots$

$d_i \neq d_{ii},\ d_i \neq 9,\ d_i \neq 0 \to \forall i\ r_i \neq r^*$

So all reals are in the list

and r^* is not in the list $\equiv F$.

So $\mathbb{R}$ is uncountable. ☒