Math 321

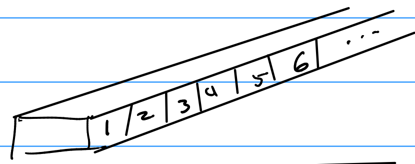
Q's — Countable sets problem

(1) to show countable you need to find the $\boxed{\text{bijection}}$ ← function

$1 \to s_1$
$2 \to s_2$
$\vdots$
← bijection

all of set S is in the list

#9 Hotel



bus 1	P_{11}	P_{12}	P_{13} ...
bus 2	P_{21}	P_{22}	P_{23} ...
bus 3	P_{31}	P_{32}	P_{33} ...
bus 4	P_{41}	P_{42}	P_{43} ...
$\vdots$			

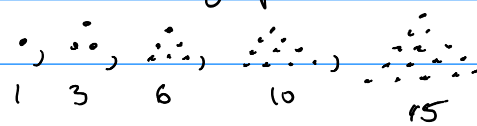
have rooms 1, 2, 3, 4, ... (already here)

new people

P_{ij} : i group #, j person #

add $i+j$

1st — 2 — [P_{11} stays in room 1
2nd — 3 — [P_{21}, P_{12} goes to 2, 3
3rd — 4 — [P_{31}, P_{22}, P_{13} goes to 4, 5, 6
4th — 5 — [$P_{41}, P_{32}, P_{23}, P_{14}$ goes to 7, 8, 9, 10
5th — $\boxed{6}$ — [$P_{51}, P_{42}, P_{33}, P_{24}, P_{15}$ goes to $\underbrace{11, 12, 13, 14, 15}_{5}$



5th Δ number

$\frac{5 \cdot 6}{2} = 15$

Bijection:

$1 \to P_{11}$ $\quad$ $5 \to P_{22}$
$2 \to P_{21}$ $\quad$ $6 \to P_{13}$
$3 \to P_{12}$ $\quad$ $\vdots$
$4 \to P_{31}$

Comes from diagonals

$P_{21,103}$ $\quad (21+103 = \underline{124}) \to 123^{rd}$ group has 123 people

last room is 123 Δ number $= \frac{123(124)}{2}$
$= 7626$

$P_{21,103}$ rooms: $\underset{102}{7504}$ — 7626 take the 103^{rd} room of block.

$\boxed{7606}$

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ex $|\mathbb{Q}| = \aleph_0$ $\mathbb{Q} = \{ \frac{a}{b} \mid a \in \mathbb{Z} \wedge b \in \mathbb{Z} \wedge b \neq 0 \wedge \text{no common factors} \}$

Consider the table

$$\begin{array}{ccccc}
0 & & & & \\
& d=1 & d=2 & d=3 & d=4 \quad d=5 \ \cdots \\
& \pm\frac{1}{1} & \pm\frac{2}{1} & \pm\frac{3}{1} & \pm\frac{4}{1} \ \cdots \\
& \pm\frac{1}{2} & \pm\frac{2}{2} & \pm\frac{3}{2} & \pm\frac{4}{2} \ \cdots \\
& \pm\frac{1}{3} & \pm\frac{2}{3} & \pm\frac{3}{3} & \pm\frac{4}{3} \ \cdots \\
& \pm\frac{1}{4} & \pm\frac{2}{4} & \pm\frac{3}{4} & \pm\frac{4}{4} \ \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{array}$$

is $\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \}$

so $\mathbb{Q}$ is here. But so are numbers like $\frac{12}{24} \notin \mathbb{Q}$

Now consider the diagonals, d, such that on $d=1$ only 0 is there. And all other $\pm\frac{a}{b}$ are on diagonal $d = a+b$

ex $+\frac{13}{17}$ is on diagonal $d = 30$, or $-\frac{7}{3}$ is on $d = 10$

So, obviously, no $\pm\frac{a}{b}$ is missed. My bijection just moves along these finite length diagonals...

$$\begin{array}{ccccc}
0 & & & & \\
& d=1 & d=2 & d=3 & d=4 \quad d=5 \ \cdots \\
& \pm\frac{1}{1} & \pm\frac{2}{1} & \pm\frac{3}{1} & \pm\frac{4}{1} \ \cdots \\
& \pm\frac{1}{2} & \pm\frac{2}{2} & \pm\frac{3}{2} & \pm\frac{4}{2} \ \cdots \\
& \pm\frac{1}{3} & \pm\frac{2}{3} & \pm\frac{3}{3} & \pm\frac{4}{3} \ \cdots \\
& \pm\frac{1}{4} & \pm\frac{2}{4} & \pm\frac{3}{4} & \pm\frac{4}{4} \ \cdots
\end{array}$$

$1 \to 0$

$2 \to +1$

$3 \to -1$

$4 \to +\frac{1}{2}$

$5 \to -\frac{1}{2}$

$6 \to +2$

$7 \to -2$

$8 \to +\frac{1}{3}$

$9 \to -\frac{1}{3}$

skip $\pm 2/2 \notin \mathbb{Q}$

$10 \to +3$

$11 \to -3$

etc

$\vdots$

Fact $\mathbb{R}$ are uncountable

pf it is enough to show reals from 0 to 1 are uncountable

assume: reals from 0 to 1 are countable. So we have a bijection..

$1 \to r_1 = 0.d_{11}\, d_{12}\, d_{13} \ldots$

$2 \to r_2 = 0.d_{21}\, d_{22}\, d_{23} \ldots$

$3 \to r_3 = 0.d_{31}\, d_{32}\, d_{33} \ldots$

$\vdots$

Note: exclude all 9 decimal expansions. Because they are a second version of a term. decimal. (ex) $0.2999\ldots = 0.3$

Now every r_i is a uniq decimal expansion.

(also bijection = one-to-one and onto says all reals between 0 and 1 are in the list)

Now consider the number $r^* = 0.d_1 d_2 d_3 \ldots$ (obviously a number between 0 and 1)

and select $d_1, d_2, d_3, \ldots$

Mark's Selection

select ① d_1 such that $d_1 \neq d_{11}, d_1 \neq 0, d_1 \neq 9$

$\xrightarrow{b/c\ \text{uniq}} r^* \neq r_1$

② d_2 such that $d_2 \neq d_{22}, d_2 \neq 0\ d_2 \neq 9$

b/c $d_2 \neq d_{22} \to r^* \neq r_2$

$\vdots$

so $\forall i\ \ d_i \neq d_{ii}, d_i \neq 0, d_i \neq 9$

$\to \forall i\ r^* \neq r_i$

r^* is not in the list

$\equiv F$ so $\mathbb{R}$ is uncountable

Book's Selection

select ① $d_1 = \begin{cases} 4 & \text{if } d_{11} \neq 4 \\ 5 & \text{if } d_{11} = 4 \end{cases}$

$\to d_1 \neq d_{11} \to r^* \neq r_1$

$\vdots$

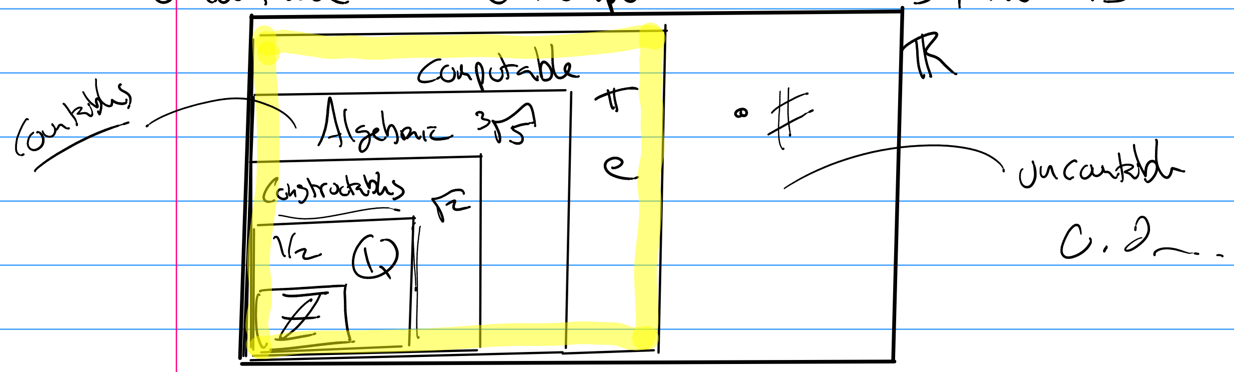
$\forall i\ \ d_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$

$\to \forall i\ \ r^* \neq r_i$

So r^* is not in the list

$\equiv F$ So $\mathbb{R}$ is uncountable.

Uncountable $\to$ Uncomputable Functions / numbers



$r_i = 0, d_{i1} d_{i2} d_{i3} \ldots$ $\qquad$ $r_n = 0.121221222122221\ldots$

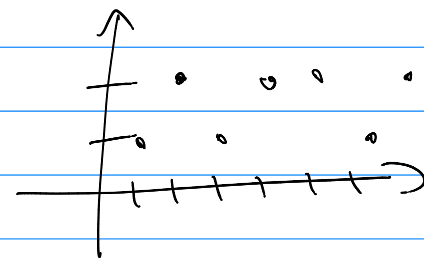
$f(n) = d_{in}$ $\qquad$ $f_n(n) = d_n$

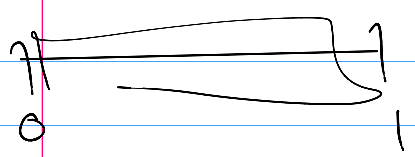
$f_n(1) = 1$

$f_n(2) = 2$

$f_n(3) = 1$

$f_n(4) = 2$





2.6 **Matrices:** (Jo)

real values

$A + B, \quad AB$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^n = A \cdot A \cdot \ldots A$$

$$A^0 = I$$

zero-one bit matrices $\qquad$ ex $\quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Exam Review 12 probs 110 pts = 100%

2.1/2.2 **Sets / Set ops** (4 probs)

① list ⇄ set-builder
↖ venn ↙

② $P(S)$? $A_1 \times A_2 \times \dots \times A_n$?

③ ④ } **ops** (set identities, membership tbls, venn diagrams, subset)

2.3 Functions (2 probs)

① give a function that is one-to-one? onto?

② **Study** #34, #35 p. 154

2.4 Seq. Sum (3 probs)

① rule for seq ⇄ seq & #'s

② Sum (show one of the $\sum k$, $\sum k^2$, $\sum k^3$) ← rules

③ **Sum** ex $\sum_{k=1}^{20} \frac{1}{k} - \frac{1}{k+1}$, $\sum_{k=1}^{40} k^2 - k + 2$, etc

2.5 Cardinality (3 probs)

(1) $\mathbb{Q}$ is countable

(2) $\mathbb{R}$ is uncountable

(3) ?, ?, ?