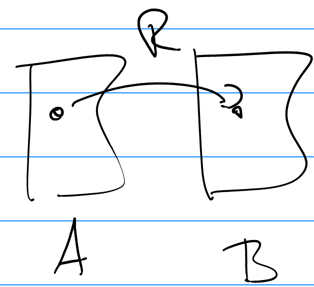
Math 322

Relation: (binary Relation) — sets of ordered pairs

R from A to B



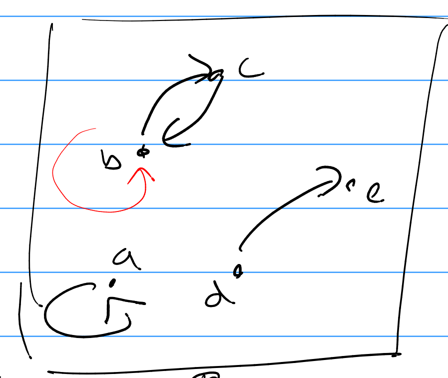
$\boxed{R \text{ on set } A}$ (from A to A)

1. reflexive $\forall e (eRe)$
2. irreflexive $\forall e (e \not R e)$
3. $\forall a \forall b (aRb \to bRa)$ symmetric
4. $\forall a \forall b (aRb \wedge bRa \to a=b)$ antisymmetric
5. $\forall a \forall b (aRb \to b \not R a)$ asymmetric

   " irreflexive and antisymmetric

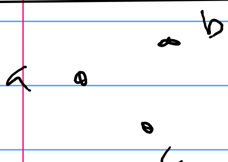
6. $\forall a \forall b \forall c (aRb \wedge bRc \to aRc)$ transitive

$bRc \wedge cRb \to \cancel{bRb}$



not ref.
not irref.
not sym.
not anti sym
not asym.
not trans.

$\boxed{R}$



$R = \{ \ \}$

not ref.
is irref.
is sym
is antisym
is asym
is trans.

ex $R = \{(a,b) \mid a, b \text{ are people and } a \text{ loves } b\}$

reflexive: $\forall a\,(aRa)$ all people love themselves

irreflexive: $\forall a\,(a \not R a)$ all people do not love themselves

symm. $\forall a \forall b\,(aRb \to bRa)$

anti-symm: $\forall a \forall b\,(aRb \land bRa \to a = b)$

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b/c $R = \boxed{\text{set}}$ of ordered pairs

so set ops can be used!

$$R_1 \cup R_2 = \{e \mid e \in R_1 \lor e \in R_2\}$$

$$= \{(a,b) \mid aR_1b \lor aR_2b\}$$

$$R_1 - R_2 = \{(a,b) \mid aR_1b \land a \not R_2 b\}$$

Composition

$$\begin{tikzcd} a \arrow[r, "R_1"] \arrow[rr, bend right, "R_2 \circ R_1"'] & b \arrow[r, "R_2"] & c \end{tikzcd}$$

Power:

$$R^1 = R$$

$$R^n = R^{n-1} \circ R$$

$R^1 = R$

$R^2 = R \circ R$

$R^3 = R^2 \circ R$

$R^4 = R^3 \circ R$

$\boxed{\text{Thm}}$ R is transitive iff $R^n \subseteq R$ $n=1,2,3,\ldots$

$\boxed{\text{Proof}}$ $\boxed{R \text{ is trans}} \to R^n \subseteq R$ $n=1,2,3,\ldots$ (case 1)

$\forall a \forall b \forall c \boxed{(aRb \wedge bRc \to aRc)}$ show $\to$ $R^n \subseteq R$ $n=1,2,3,\ldots$

assume

try induction.

Basis: $\boxed{\text{[illegible]}}$

Inductive: $R^k \subseteq R \to R^{k+1} \subseteq R$

$R^k \circ R$

R — R^{k+1} — R

case 2 $\left[R^n \subseteq R \quad n=1,2,\ldots \to R \text{ is trans} \right]$

univ. inst. $(n=2)$

$R^2 \subseteq R$ is $(R \circ R) \subseteq R$ is $aRb \wedge bRc \to aRc$

a c

Representations of R

1. list pairs
2. tally tables
3. zero-one matrix

$R = \{(a,a), (a,c), (b,c), (c,c)\}$

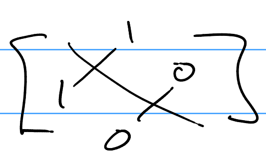
on $A = \{a, b, c, d\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

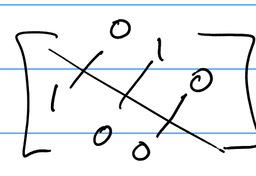
a) ref? $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

b) irref? $\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$

c) sym?



d) antisym



Operations $R_1 \cup R_2$: $M_{R_1} \vee M_{R_2} = M_{R_1 \cup R_2}$

$R_1 \cap R_2$: $M_{R_1} \wedge M_R = M_{R_1 \cap R_2}$

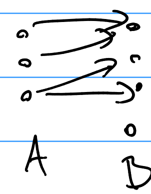
$R_1 \circ R_2$: $M_{R_2} \odot M_{R_1} = M_{R_1 \circ R_2}$

R^n : $M_R \odot M_R \odot \cdots \odot M_R = M_R^{[n]} = M_{R^n}$

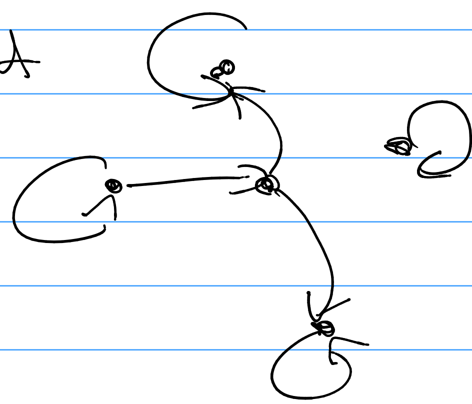
Note: R is trans iff $R^n \subseteq R$ $n = 1, 2, \ldots$

$M_{R^2} = \underline{M_R^{[2]}}$ $\qquad M_R$

(4) arrow diagram:



(5) if R is on A



digraph

directed graph

For the usual properties of reflexive, symmetric, antisym., transitive

(See video)