Math 322

Q's 9.1 #7a) $R = \{(x,y) \mid x \neq y\}$ on A = Integers

ref? $\forall a (aRa)$ for all ints a, $a \neq a$ counter example $0=0$ so false

sym? $\forall a \forall b (aRb \to bRa)$ true

related means not equal

anti sym? $\forall a \forall b (aRb \wedge bRa \to a=b)$ counter example $0, 1$ False

trans? $\forall a \forall b \forall c (aRb \wedge bRc \to aRc)$

1 2 2 1 1 1 ← counter example false.

6e) aRb means $a \cdot b \geq 0$ $-3, [-2, -1, 0, 1, 2], 3$

ref? $\forall a (aRa)$ true $a \cdot b \geq 0$

sym? $\forall a \forall b (aRb \to bRa)$

$a \cdot b \geq 0$ $ba \geq 0$

trans? $\boxed{aRb \wedge bRc \to aRc}$ ←

$a \cdot b \geq 0$ $b \cdot c \geq 0$ $a \cdot c \geq 0$

$3 \cdot 0$ $0 \cdot -4$

counter example

9.4 Closures (making new relations that a property of interest)

Consider R is it?

1. reflexive? aRa No? Some aRa
2. sym.? $aRb \to bRa$ Some $a \cdot \to \cdot b$
3. anti sym? $aRb \wedge bRc \to a=b$ Some $a \cdot \rightleftarrows \cdot b$
4. trans? $aRb \wedge bRc \to aRc$ $a \cdot \to \cdot b \to \cdot c$

For reflexive, sym., transitive if they have counter example it's because they are missing specific ordered pairs.

Idea: Make a new relation, $\hat{R}$ such that

closure

① $R \subseteq \hat{R}$ (add ordered pairs to R)

② $\hat{R}$ has my property of interest

③ added the least number of ordered pairs possible.

① Reflexive Closure

ex $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so, not reflexive.

Def $\Delta = \{ (a,a) \mid a \in A \}$ $M_\Delta = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ Identity matrix

Reflexive Closure $R \cup \Delta$: $M_R \vee M_\Delta$

ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$ ← ref. closure

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Sym. Closure

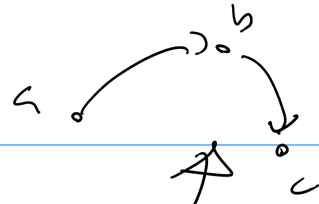
Note: $R^{-1} = \{ (b,a) \mid aRb \}$

$$M_{R^{-1}} = M_R^T$$

$R \cup R^{-1}$: $M_R \vee M_R^T$

ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$ sym. closure

Transitive Closure?

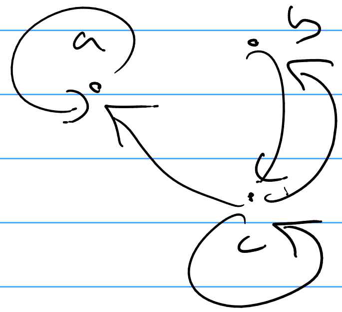


$aRb \wedge bRc \to a ? c$

① find all of these and add (join) aRc

② remember R is trans iff $\forall n \; R^n \subseteq R$

ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$



path $(b,c),(c,c),(c,b),(b,c)$
$(c,a),(a,a)$

length = 6

shorthand b,c,a

Def a path from x_0 to x_n is a seq of edges

Path $\to \underbrace{(x_0,x_1)}_{e_1}, \underbrace{(x_1,x_2)}_{e_2}, \underbrace{(x_2,x_3)}_{e_3}, \ldots, \underbrace{(x_{n-1},x_n)}_{e_n}$

length = # of edges

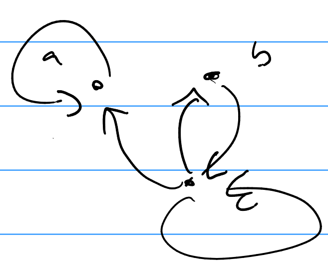
vertex version: $x_0, x_1, x_2, x_3, \ldots, x_n$

if $x_0 = x_n \to$ call it a circuit.
if every edge you see in path occurs once $\to$ simple

paths and transitivity? R^n

thm $(a,b) \in R^n$ iff a path of length n from a to b.

ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$



c, b, c, a so $(c,a) \in R^3$

$$R^{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \boxed{1} \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Consider: $R \cup R^{2} \cup R^{3} \cup R^{4} \cup \dots = \bigcup_{n=1}^{\infty} R^{n} = R^{*}$

Def: $R^{*} = \bigcup_{n=1}^{\infty} R^{n}$ connectivity relation

Note: $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \dots$

Th^m R^* is the transitive closure.

(consider R is trans iff $\forall n \; R^n \subseteq R$)

We know trans. closure exists (R^*)

Issue is ∞ operations! $M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \dots$

Hope: Pigeonhole Principle

Consider: R on set A, $|A| = m$ $a_1, a_2, a_3, \dots a_m$

any path: $x_0, x_1, x_2, x_3, \dots, x_n$ (length is n)

(See video)

Th^m $M_{R^*} = M_R \vee M_R^{[2]} \vee \dots \vee M_R^{[n]} \vee \left[M_R^{[n+1]} \vee M_R^{[n+2]} \vee \dots \right.$

nothing new

has a shorter path

$|A| = n$

So $$\boxed{M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}}$$

ex

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ find $M_{R^2} = M_R^{[2]}$ $M_{R^3} = M_R^{[3]}$

$$\boxed{M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}}$$

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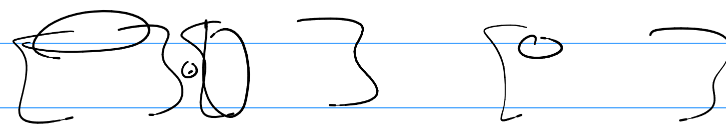
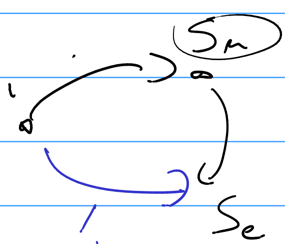
this is <u>slow</u>!



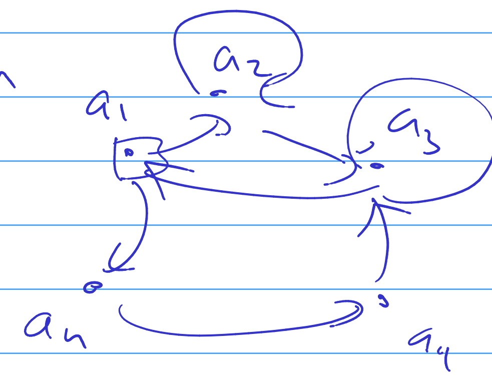
10x10 10x10

<u>Is there a faster technique?</u> Warshal's Algorithm.

<u>Idea:</u>



if $|A| = n$



ex

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 1 & 1 \end{array}$$

a goes to ▢

▢ goes to a