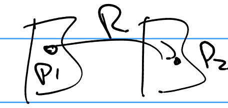
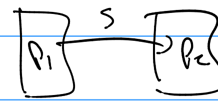
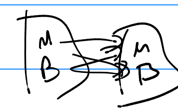
Math 322

Q's 9.1 #33 $R = \{(a,b) \mid a \text{ is a parent of } b\}$

$S = \{(a,b) \mid a \text{ and } b \text{ are siblings}\}$

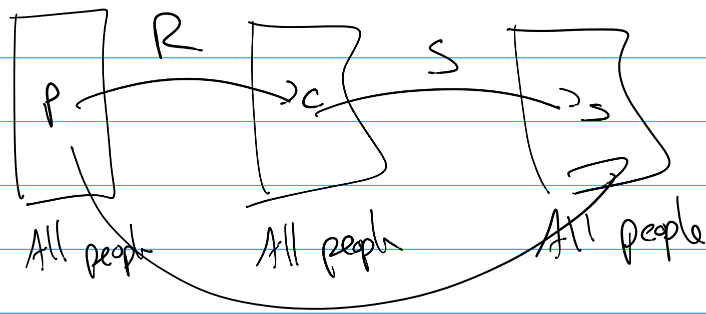






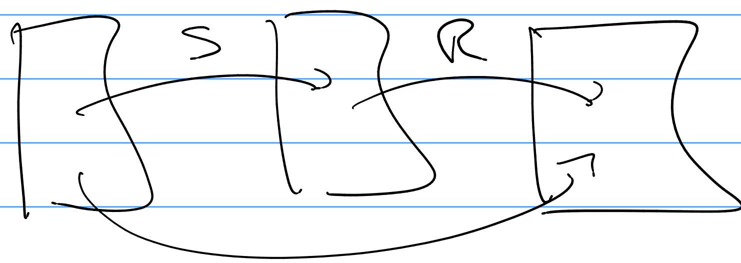
$S \circ R = ?$ $R \circ S = ?$

$S(R(p))$



$S \circ R$

$R \circ S$



$R \circ S$

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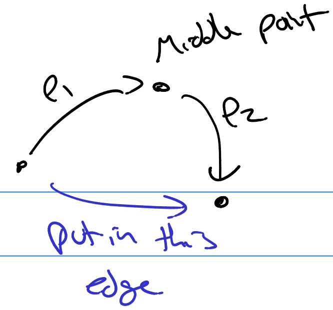
given R on set A, M_R, $|A| = n$

1. to create reflexive closure $M_R \vee I$
2. to create sym. closure $M_R \vee M_R^T$
3. to create trans closure

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

Warshal's Algorithm



$$M_R = \begin{array}{c|c} & e_1 \quad e_i \quad e_n \\ \hline \begin{array}{c} e_1 \\ \vdots \\ e_i \\ \\ e_n \end{array} & \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix} \end{array}$$

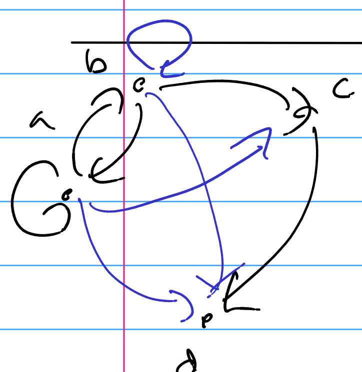
i^{th} row: e_i goes to $\boxed{A}$

i^{th} col: $\boxed{A}$ goes to e_i

ex

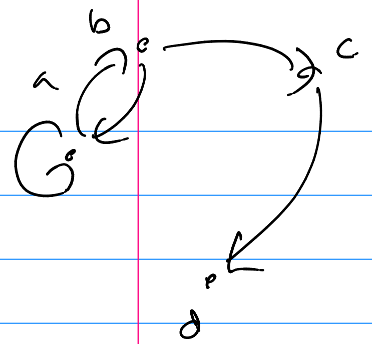
$$\begin{array}{c|c} & a \quad b \quad c \quad d \\ \hline \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & & 1 \\ & 0 & & \end{bmatrix} \end{array}$$

$M_R = W_0 =$ [ ] row 1, col 1 $\xrightarrow{\text{Make new}}$ $W_1 =$ [ ] row 2, col 2 $\longrightarrow$ $W_{n-1} =$ [ ] $\longrightarrow$ M_{R^*}



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad W_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad W_4 = M_{R^*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]}$$

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ b \\ \\ \\ \end{array} \overset{\qquad\qquad\quad a}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \leftarrow$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

etc ...

9.5 R on set A is reflexive, symmetric, and transitive. we call R an <u>equivalence relation</u>.

<u>Notation:</u> If you show (or "someone" has) R to be reflexive, symmetric and transitive we can replace the symbol "R" with "$\sim$"

<u>before:</u> $(a,b) \in R$ $\xrightarrow{\text{now}}$ $(a,b) \in \sim$

aRb ↑ related to $\qquad$ $a \sim b$ ↑ equivalent to

(ex) $R = \{(a,b) \mid a,b \in \mathbb{R} \text{ and } a=b\}$

Is it an equiv. relation?

① <u>reflexive?</u> $\forall a\,(aRa)$ means $\forall a\,(a=a)$ <u>true</u>

② <u>sym?</u> $\forall a \forall b\,(aRb \to bRa)$ means $(a=b \to b=a)$ <u>true</u>

③ <u>trans?</u> $(aRb \wedge bRc \to aRc)$ means $(a=b \wedge b=c \to a=c)$ <u>true</u>

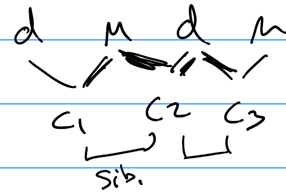
so $R = \{(a,b) \mid a = b\}$ is an equiv. relation

(ex) $S = \{(a,b) \mid a \text{ and } b \text{ are siblings}\}$

① ref? $\forall a\,(aSa)$ means "a and a are siblings" true

② sym? $aSb \to bSa$ means "a and b are siblings $\to$ b and a are siblings" true

③ trans? $aSb \wedge bSc \to aSc$



No counter example.

(ex) $R = \{(a,b) \mid a \equiv_n b\}$ on $A = \mathbb{Z}$

(ex) $\boxed{n=5}$ $\ldots \equiv_5 -5 \equiv_5 0 \equiv_5 5 \equiv_5 10 \equiv_5 15 \ldots$ $(0,5)$ $(10,0)$ $(-5,15)$

$\ldots \equiv -8 \equiv_5 -3 \equiv_5 2 \equiv_5 7 \equiv_5 12 \equiv_5 17 \ldots$

$(7,2)$ $(-8,17)$

Equiv?

① ref? $\forall a\,(aRa)$ means $a \equiv_n a$ true

② sym? $aRb \to bRa$ means $a \equiv_n b \to b \equiv_n a$ true

③ tran? $aRb \wedge bRc \to aRc$ means

$a \equiv_n b \wedge b \equiv_n c \to a \equiv_n c$

true.

So $R = \{(a,b) \mid a \equiv_n b\}$ is an equiv. relation

so $(n=5)$ $0 \sim 5 \sim 10 \sim -5 \sim 25 \sim \ldots$

$1 \sim 6 \sim -4 \sim$

ex $\left(6^{121} + 5^{4^{1000}}\right)^{21} \bmod 5$

$= \left(1^{121} + 0^{4^{100}}\right)^{21} \bmod 5 = \boxed{1}$

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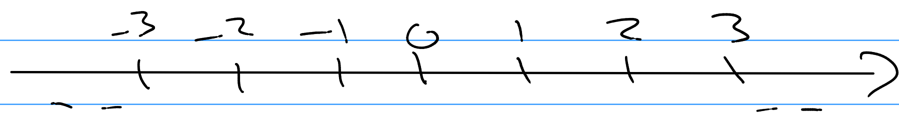
b/c equiv. relations allow us to find equiv. objects

→ Collect them. (R is an equiv. relation)

Def $[a]_R = \{ e \mid aRe \}$

Note: if R is "known" use $[a]$

ex $a \equiv_5 b$



$[0] = \{\ldots, -10, -5, 0, 5, 10, \ldots\} = [5]$
$[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\} = [6]$
$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$
$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$

Def Partition: $S_1, S_2, S_3, \ldots, S_n$ & A has

1. $A = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n$
2. $S_i \neq \emptyset$
3. $S_i \cap S_j = \emptyset$ if $i \neq j$

Thm equiv. classes of a set form a partition of the set.
and given any partition there is an equiv. relation on the set whose equiv. classes equal the partition.

Thm R is an equiv. relation on A. And the following are logically equivalent.

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] \neq \phi$