Math 322

How to Study for Exam

1. don't cram. Monday Night = bad!
2. 1-2 hrs each day.
3. coffee
4. rest

9.6 Partial Orderings

if R on a set S is reflexive, antisymmetric, and transitive we will call it a partial ordering.

ex $R = \{(a,b) \mid a \le b,\ a,b \in \mathbb{Z}\}$ $\qquad a < b$

ref? aRa means $a \le a$ true.

anti-sym? $aRb \wedge bRa \to a = b$ means: $a \le b \wedge b \le a \to a = b$ true

trans? $aRb \wedge bRc \to aRc$ means: $a \le b \wedge b \le c \to a \le c$ true

So $\le$ is a partial ordering of $\mathbb{Z}$

Def: (1) if R on set S is a partial ordering (ref, antisym, trans) use "$\preccurlyeq$" for R.

(2) put $\preccurlyeq$ with its set S we call S a partially ordered set use $(S, \preccurlyeq)$ call it a poset

ex's $(\mathbb{Z}, \leq)$

ex $R = \{(a,b) \mid a \mid b\}$ on $\mathbb{Z}^+$

such that

number theory divides

$a \mid b$ means $a \cdot m = b$

partial order?

(1) ref. aRa means $a \mid a$ $\boxed{\text{true}}$ b/c $a \cdot 1 = a$

(2) anti sym $aRb \land bRa \to a = b$

means: $a \mid b \land b \mid a \equiv (a \cdot m_1 = b) \land b \cdot m_2 = a$

$\equiv (b \cdot m_1 \cdot m_2 = b)$

so $m_1 = m_2 = 1$ $\therefore a = b$

(3) trans: $aRb \land bRc \to aRc$

means $a \mid b \land b \mid c \to a \mid c$ (do at home)

so $\boxed{(\mathbb{Z}^+, \mid) \text{ is a poset}}$ true

it allows ordering: ex $2 \preceq 2$ $2 \prec 4$ $2 \prec 6$

Def: if $a \preceq b$ or $b \preceq a$ we call a, b comparable.

what about $2 \prec 6$ $3 \prec 6$ but 2 vs 3

but $2 \not\preceq 3$ and $3 \not\preceq 2$

2, 3 do not compare (incomparable)

→ this is why we use the word $\boxed{\text{partial.}}$

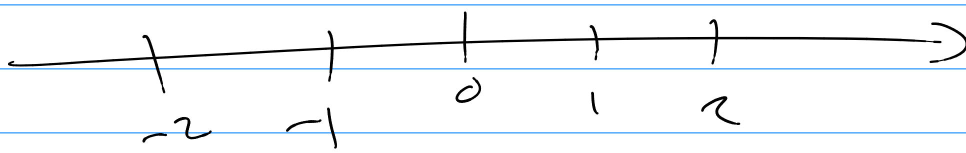
$\boxed{\text{Def}}$ if $(S, \preceq)$ is a poset

$\boxed{\text{and}}$ every a, b in S compare

call $(S, \preceq)$ a total order

ex $(\mathbb{Z}^+, |)$ is a poset but $\boxed{\text{not}}$ a total order

$(\mathbb{Z}, \leq)$ is a total order



$\boxed{\text{Def:}}$ if $(S, \preceq)$ is a total order

$\boxed{\text{and}}$ for any subset of S there is a least element.

call $(S, \preceq)$ well ordered.

ex $(\mathbb{Z}, \leq)$ is <u>not</u> well ordered

$(\mathbb{Z}^+, \leq)$ <u>is</u> well ordered

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ex <u>lexicograph sorts</u>

$S = (\text{n-tuples})$ $\quad aRb$ if $a = (a_1, a_2, \ldots, a_n)$

$b = (b_1, b_2, \ldots, b_n)$

$a \preceq b$ <u>when</u> $a_1 < b_1$

or if $a_1 = b_1$ $\quad a_2 < b_2$

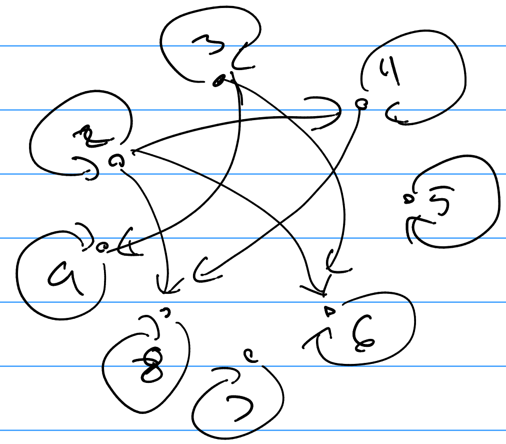
$\vdots$

$a_i = b_i$ for all i $\quad a = b$

Visualization: digraphs: $a \preceq b$ $\quad a \to b$

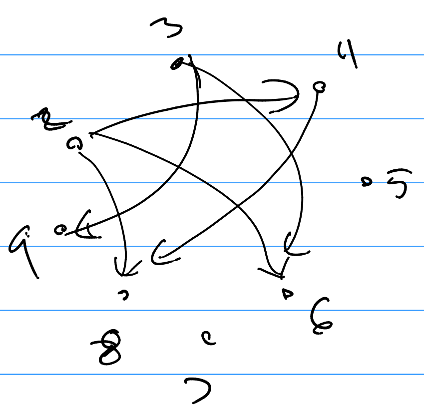
ex $(\{2,3,4,5,6,7,8,9\}, |)$

~~digraph~~



b/c $(S, |)$ is a poset it $\boxed{\text{is}}$ reflexive.

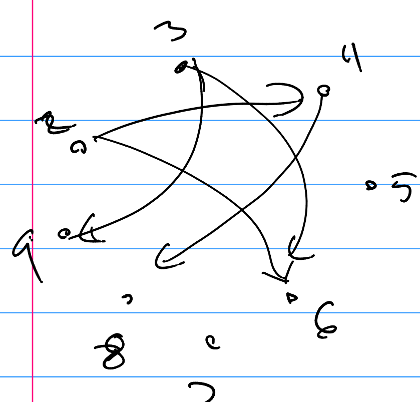
So don't draw loops to make it look better



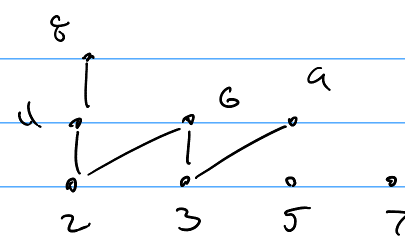
b/c $\boxed{\text{trans}}$

$2 \preceq 4 \wedge 4 \preceq 8 \to 2 \preceq 8$

remove all trans. edges (to make it look better)



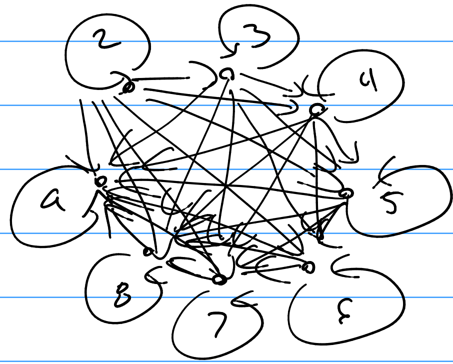
Make arrows point up and remove direction.

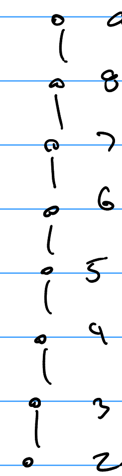


Hasse diagram

(ex) $(\{2,3,4,5,6,7,8,9\}, \leq)$

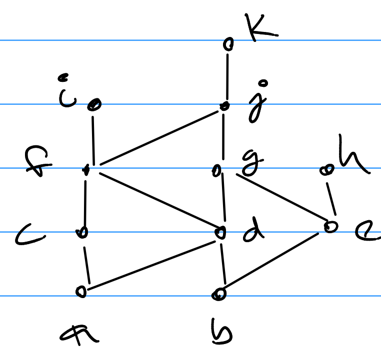
digraph





Hasse diagram

(ex) Hasse Diagram



$e_1 \preceq e_2$

upwards path from e_1 to e_2

(ex) $c \preceq f$ $\quad c \not\preceq g$

$c \preceq j$

etc

So c compares to $\{a, c, f, i, j, k\}$

c is incomparable to $\{b, d, e, g, h\}$

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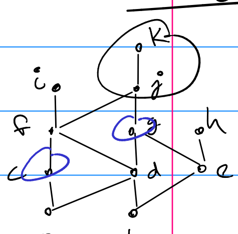
topological features ("tops", "bottoms")



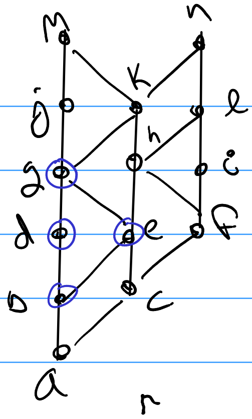
ex

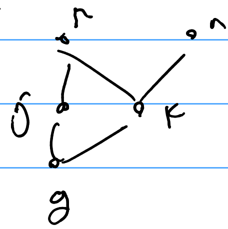
1. Maximal element(s): elements with no one above $\{i, k, h\}$
2. Minimal element(s): elements with no one below $\{a, b\}$
3. greatest element: element above all others $\{\ \}$
4. least element: element below all others $\{\ \}$

given a subset of S what are the ... Subset $= \{c, g\}$

5. upper bounds of subset: set of all elements above $\{j, k\}$
6. lower bounds of subset; set of all elements below $\{a\}$
7. least upper bound $\{j\}$
8. greatest lower bound $\{a\}$

(ex)





Maxs : $\{m, n\}$ Mins : $\{a\}$

greatest : $\{\}$ least : $\{a\}$

Subset = $\{b, d, e, g\}$

lower bounds = $\{a, b\}$

upper bounds = $\{g, j, k, m, n\}$

greatest lower bound = $\{b\}$

least upper bound = $\{g\}$

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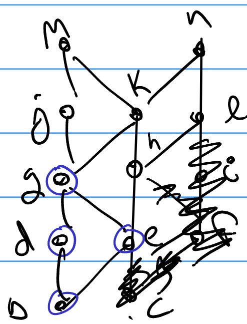
given a poset can you add edges so that the new relation is a total ordering?

Topological Sort

take minimals from right to left.



$a < c, f, i, b$