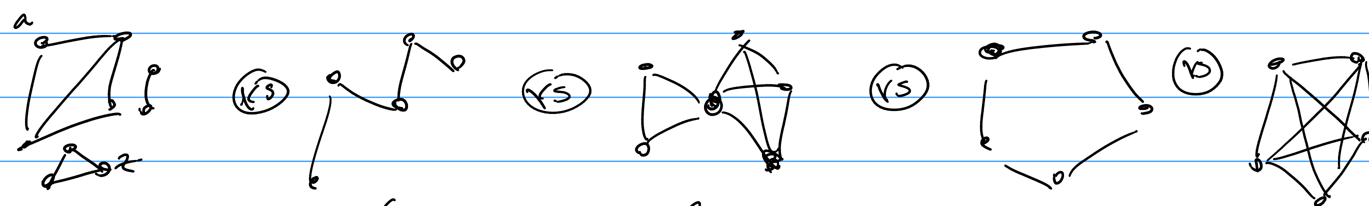
Math 322

Consider:



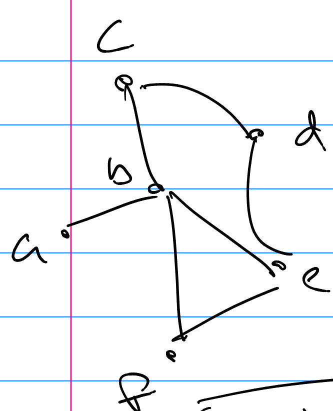
G_1 $\qquad$ G_2 $\qquad$ G_3 $\qquad$ C_5 $\qquad$ K_5

Def Connectedness

1st a path is a seq of edges; $e_1, e_2, e_3, \dots, e_n$ in $G=(V,E)$.

- length $= n$
- $n \geq 0$
- if there is no confusion on mult. edges we can use $x_0, x_1, x_2, \dots, x_n$ (the vertices in edges) to represent the path.

- $x_0 = x_n$ call path a circuit
- if each e_i in path is unique we call path simple.



path: a, b, c, b, e — length = 4, not simple, not a circuit

path: b, c, d, e, b — length = 4, is simple, is circuit

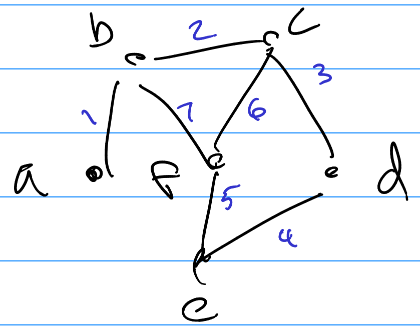
(undirected)

So $G=(V,E)$ is connected if there is a path between every distinct pair of vertices

- If G is not connected → call it disconnected
- We say we disconnect a graph if you can remove vertices and/or edges and the subgraph produced is disconnected.

Th^m For every pair of distinct points in a connected graph there is a simple path between them.

Idea (ex)



a, b, f, b, c
1 7 7 2

a, b, c

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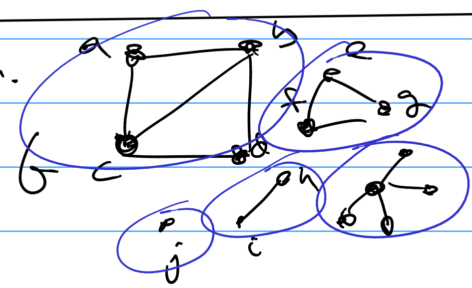
on disconnected graphs:

(ex)



→ connected components
{a, b, c, d}
{e, f, g}
{j}
etc

Def: the connected components of a graph is a maximal connected subgraph of it.

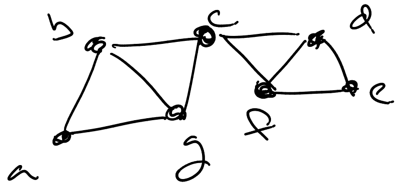
Unconnected graphs have 2 or more connected components

How vulnerable is G to being disconnected?

(1) Focus on vertices (remove a vertex and all its inc. edges)

Def: A vertex cut is a set of vertices such that when you remove them you disconnect G.

(ex)



Vertex cuts: $\{d, f\}$

$\{b, g\}$

$\{c\}$

Def: Vertex Connectivity : $\kappa(G) = n$, is the least number of vertices to disconnect the graph.

find smallest vertex cut.

Def: $\kappa(G) = 1$ the one vertex is called the cut vertex.

For any G

$$0 \le \kappa(G) \le n-1$$

G was already disconnected

only for K_n

Note: Most connected is K_n

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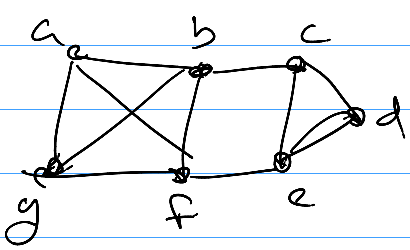
② Focus on edges (how many edges to cut to make G disconnect)

Def: set of edges that disconnect a graph is an edge cut.

(ex)



edge cuts: $\{\{b,c\}, \{f,e\}\}$

$\{\{g,a\}, \{g,b\}, \{g,f\}\}$

etc

= Def edge connectivity : $\lambda(G) = n$, least number of edges to disconnect G

= Def $\lambda(G) = 1$, the one edge is called the cut edge.

$\lambda(G) = 2$ $\kappa(G) = 2$ — ex $\{b, f\}$

disconnected already

$$0 \le \lambda(G) \le n-1$$

only for K_n

Thm $0 \le \kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$

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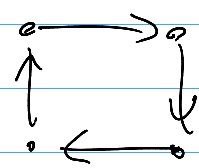
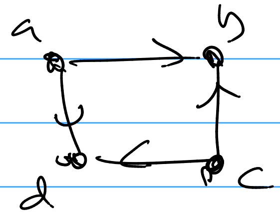
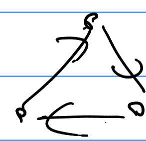
directed graphs

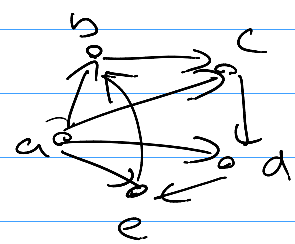
Def Strongly connected G has path to and from all distinct pairs of vertices.

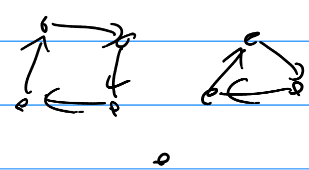
ex  vs 

Def Weakly connected G if underlying undirected graph is connected.

so directed graphs can be..

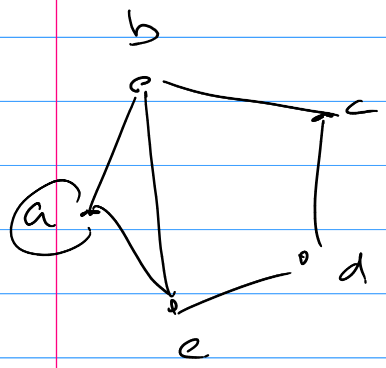
① strong and weak 

② not strong, and is weak 

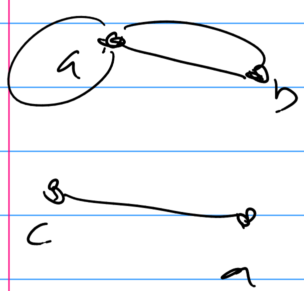
③ not strong, not weak 

A_G is G's adj. matrix

$\boxed{\text{Th}^n}$ $A_G^r = [a_{ij}]$ a_{ij} = # of different paths from v_i to v_j



$$A_G = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad A_G^2 = \begin{matrix} 3 & 2 & 1 & 1 & 2 \\ 2 & 3 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 & 3 \end{matrix}$$



$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$