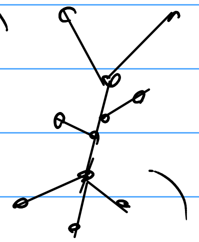
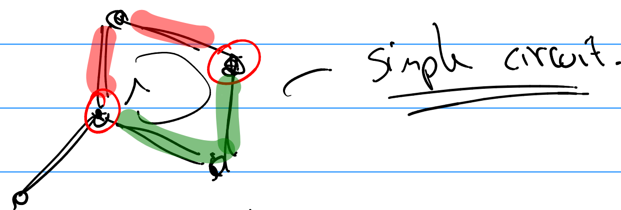
Math 322

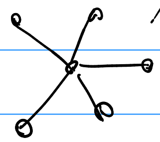
Trees

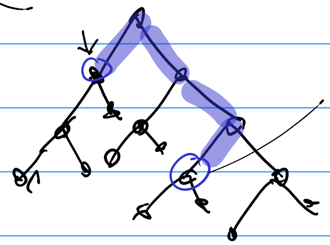




"tree"

not a "tree"





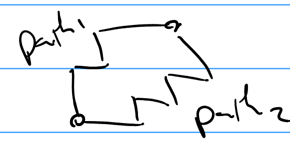
Def: G is a tree if it is a connected undirected graph with no simple circuits.

Thm: G is a tree if and only if there is a uniq. simple path between any two vertices.

Pf case 1 (tree → uniq. simple path)

contrapositive: $\neg$ uniq → $\neg$ tree

if $\neg$ uniq. simple path says between some two vertices there are 2 or more simple paths.

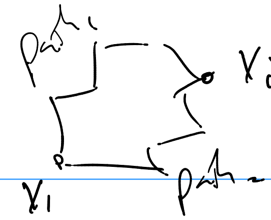


when you follow both paths you have a simple circuit
→ $\neg$ tree.

Case 2 (uniq simple path → tree)

contrapositive? $\neg$ tree → $\neg$ uniq. simple path

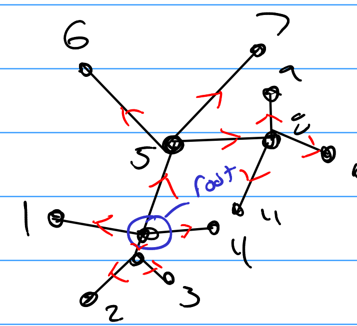
→ tree says we have a simple circuit that has another vertex in it.



implies non-uniq simple paths between v_1 and v_i.

Rooted Tree

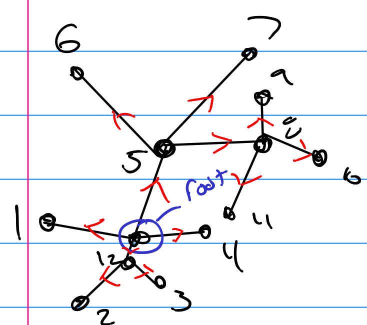
tree (undirected)

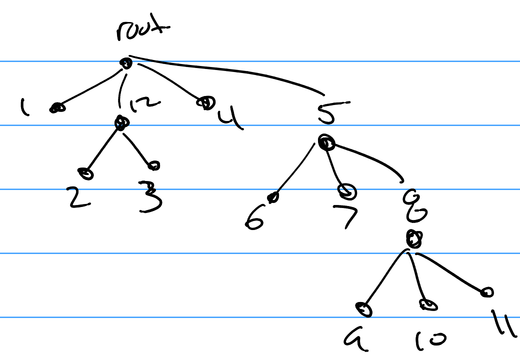


① to make a rooted tree pick one vertex as "root"

② b/c uniq. simple path between any two vertices → uniq path from root to any vertex.

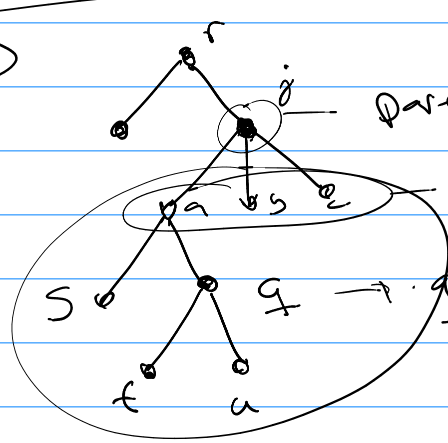
③ direction is 'away' from root.





direction is down ↓

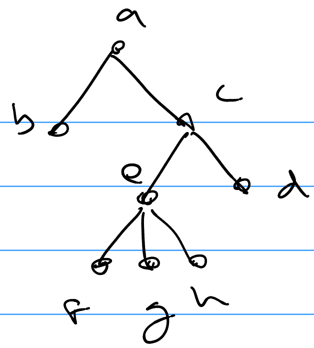
terms



parent of a, b, c

children of j

→ a, j, r are ansc. of g



internal vertex = have at least one child

ex $i = \{a, c, e\}$

leaf = have no children

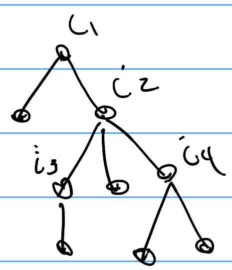
ex $l = \{b, d, f, g, h\}$

total vertices $\boxed{n = i + l}$

M-ary rooted tree

→ each internal vertex has at most M children

ex



3-ary tree

$i = 4$

$l = 5$

$n = 9$

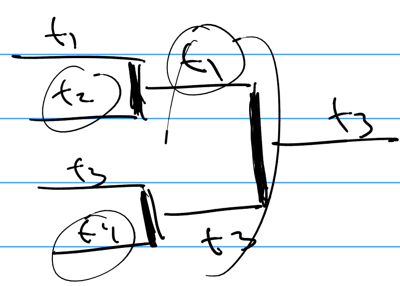
Note: each parent has a uniq. simple path (single edge) to their child.

So $\underline{(n-1)}$ children (all − root)

not a child

$|E| = n - 1$

ex



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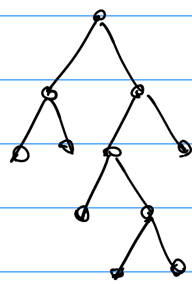
full m-ary rooted tree

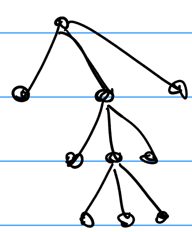
each internal vertex has exactly m children.

(ex) full 2-ary (binary) rooted tree



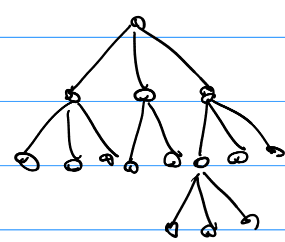
balanced m-ary tree

all leaves are at bottom or one above bottom.



not balanced

full 3-ary



balanced

3-ary

def.

level = 1

level = 2

level = 3

height = 3

Useful th^m's $|V| = n$ $|E| = n-1$ $n = i + l$

th^m full m-ary tree with i internal vertices

$$n = \underbrace{mi}_{\text{children}} + \underbrace{1}_{\text{root}}$$

Thm #4 p.753 full m-ary tree with

① n vertices, then $i = \frac{n-1}{m}$, $l = \frac{(m-1)n+1}{m}$

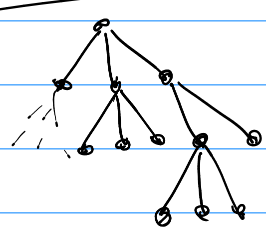
② i internal, then $n = mi+1$, $l = (m-1)i+1$

③

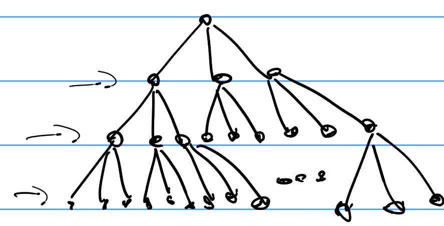
$\boxed{n = i + l \qquad n = mi+1}$ ← know these two things

ex full 7-ary tree $\quad n = i + l$
$\quad n = 7i + 1$

① $n = 70$
$$\begin{cases} 70 = i + l \\ 70 = 7i + 1 \end{cases}$$



$l = 8$



$l = 27$

full 3-ary balanced

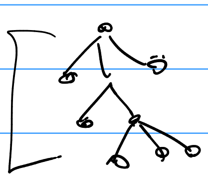
Thm $l \le m^h$, h = height of tree $(h = 0, 1, 2, 3 \ldots)$

Pf (induction)

Basis: $[h=0]$ tree: $\cdot$ root $\quad l = 1$
$m^0 = 1$
$l \le m^0 \to 1 \le 1$ true

Inductive: assume @ $h=k$ $\qquad$ show @ $h = k+1$

$h=k$

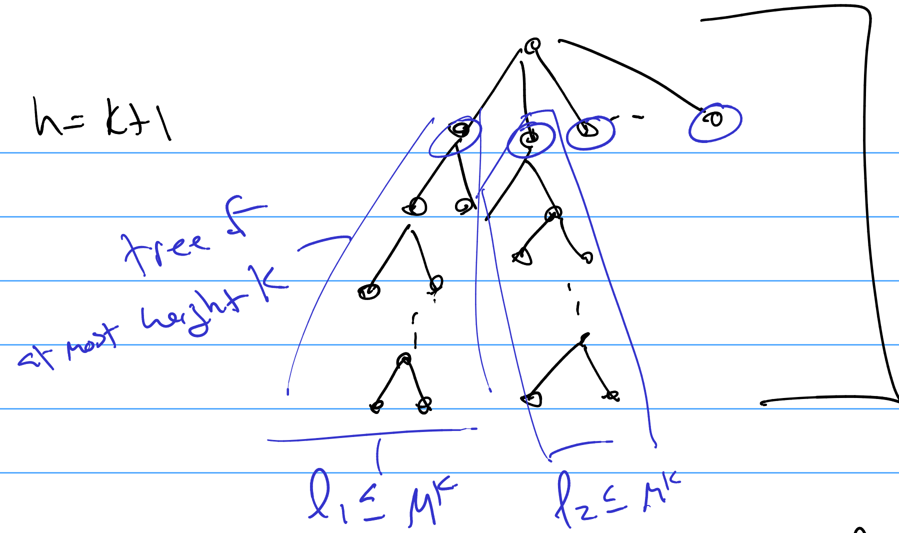


$l \le m^k$

$h = k+1$

$l \le m^{(k+1)}$

tree of $h = k+1$



height $k+1$ has at most

$$l \le l_1 + l_2 + \dots + l_m$$
$$l \le m^k + m^k + \dots + m^k$$
$$l \le m^{k+1}$$

$$l \le m^h \Rightarrow \log_m l \le h$$

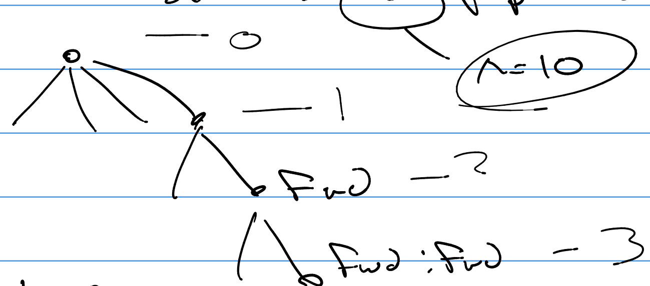
Thm $h \ge \lceil \log_m l \rceil$

and if full and balanced $h = \lceil \log_m l \rceil$

(ex) text: $\boxed{\text{Fwd ; Fwd ; Fwd ; Fwd}}$ 

send to 10 people or ...

$h = 5$ $m = 10$ full 10-ary



$\boxed{l \le m^h}$

$l \le 10^5 = 100000$

$n = i + l$

i: sent text

l: Do not send text

$|E| = n - 1$ all children

worst case → <u>balanced</u> $l = 100{,}000$

$n = 10 \quad h = 5 \quad l = 100{,}000$ → $n = ?$, $i = ?$

$$\left.\begin{aligned} n &= i + l \\ n &= mi + 1 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} n &= i + 100{,}000 \\ n &= 10i + 1 \end{aligned}\right.$$

$$0 = 9i - 99{,}999$$

$$i = 11{,}111$$

$n = 111{,}111$
$i = 11{,}111$
$l = 100{,}000$

edge = text $0.10 per person → $0.20 / text

|edges| = 111,110 texts go out

<u>Cost</u> <u>$22,222</u>