Math 322

Boolean Algebra:

objects: set $B = \{0, 1\}$ bits $= \{0, 1\}$

2 objects in → 1 object out

rules: a) two binary operators $\wedge, \vee$ object $\wedge$ object, object $\vee$ object.

b) one unary operator $\overline{\square}$ object

boolean variable x is 0 or 1

typically ↳ define ops

x
1
0

x	y	$x \wedge y$	$x \vee y$	$\overline{x}$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

check laws $\overline{\overline{x}} = x$

double negation law

x	$\overline{x}$	$\overline{(\overline{x})}$
1	0	1
0	1	0

equal

$x \vee x = x$ $x \wedge x = x$ idempotent

$x \vee \overline{x} = 1$

$x \wedge \overline{x} = 0$

x	$\overline{x}$	$x \vee \overline{x}$
1	0	1
0	1	1

$\overline{x \wedge y} = \overline{x} \vee \overline{y}$ Demorgan's

x	y	$x \wedge y$	$\overline{x \wedge y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \vee \overline{y}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

equal

Minimal facts to have a Boolean Algebra.

objects: $B = \{0, 1\}$

ops: $\wedge, \vee, \overline{B}$

and the following hold:

1. $b \vee 0 = b$, $b \wedge 1 = b$ — Identity laws
2. $b \vee \overline{b} = 1$, $b \wedge \overline{b} = 0$ — Complement laws
3. $(b_1 \wedge b_2) \wedge b_3 = b_1 \wedge (b_2 \wedge b_3)$, $(b_1 \vee b_2) \vee b_3 = b_1 \vee (b_2 \vee b_3)$ — Associative laws
4. $b_1 \wedge b_2 = b_2 \wedge b_1$, $b_1 \vee b_2 = b_2 \vee b_1$ — commutative laws
5. distributive laws: $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$

$$b_1 \vee (b_2 \wedge b_3) = (b_1 \vee b_2) \wedge (b_1 \vee b_3)$$

So $b \wedge b = b$ (idempotent) could be shown by only using above 5.

Start with $b = b \wedge 1$ (by identity)

$= b \wedge (b \vee \overline{b})$ (by comp. law)

$= (b \wedge b) \vee (b \wedge \overline{b})$ (by distrib.)

$= (b \wedge b) \vee (0)$ (by comp. law)

$= b \wedge b$ (by identity)

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objects = 0, 1
ops = +, $\circ$, $\overline{\square}$

back to typical . . . (digital design)

bit-table

x	y	$\overline{x}$	$x+y$	$x \circ y$
1	1	0	1	1
1	0	0	1	0
0	1	1	1	0
0	0	1	0	0

use bit tables to verify all the laws.

Boolean Expression: combination of bits, variables, ops.

ex ① $x + [\overline{(y \circ z)} \circ 1] + \overline{0}$

② $x + \overline{y}$

③ $(\overline{x+y}) \circ (\overline{x} + 0)$

bit tables:

x	y	$\overline{x}$	0	$\overline{x}+0$	$x+y$	$\overline{x+y}$	$(\overline{x+y}) \circ (\overline{x}+0)$
1	1	0	0	0	1	0	0
1	0	0	0	0	1	0	0
0	1	1	0	1	1	0	0
0	0	1	0	1	0	1	1

(x, y) expression

$(1,1) \to (0)$

$(1,0) \to (0)$

$(0,1) \to (0)$

$(0,0) \to (1)$

function taking (x,y) ordered pairs and return a bit.

Boolean Function: B^n : (n-tuple of n-bits) $\to$ (bit)

$f(b_1, b_2, \ldots, b_n)$ = expression of b_i boolean variables

boolean function of degree n.

ex $f(x,y,z) = \bar{x} + ((y \cdot \bar{z}) + \bar{y})$

x	y	z	$\bar{x}$	$\bar{y}$	$\bar{z}$	$y \cdot \bar{z}$	$(y \cdot \bar{z}) + \bar{y}$	$\bar{x} + ((y \cdot \bar{z}) + \bar{y})$
1	1	1	0	0	0	0	0	0
1	1	0	0	0	1	1	1	1
1	0	1	0	1	0	0	1	1
1	0	0	0	1	1	0	1	1
0	1	1	1	0	0	0	0	1
0	1	0	1	0	1	1	1	1
0	0	1	1	1	0	0	1	1
0	0	0	1	1	1	0	1	1

$(1,1,1) \xrightarrow{f} (0)$

$(1,1,0) \longrightarrow (1)$

$(1,0,1) \longrightarrow (1)$

$\vdots$

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f = expression (finite ops $\pm$, finite (countable) variables)

have countably infinite possible expressions.

ex $f(x,y)$ has countably infinite possible expressions

$f(x,y) = 1$, $f(x,y) = \left(\overline{\overline{\bar{x} + \bar{y}} + 0}\right) \cdot 1$, ...

$\boxed{\text{but}}$ all of them are functions

have a table

x	y	f
1	1	← 1 or 0
1	0	← 1 or 0
0	1	← 1 or 0
0	0	← 1 or 0

$|\text{tables}| = 2^{|\text{rows}|} = 2^4 = 16$

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f is B^n (n-tuples input)

$|\text{tables}| = 2^{|\text{rows}|} = 2^{2^{|\text{vars}|}} = 2^{2^n}$ (finite)

[ex] **Program = ?** $\quad f: B^n \to B$

only 2^{2^n} total "exist",

So... if only 2^{2^n} exist (unique boolean tables)

Can we find a "**best**" expression for a given table.

(ex)

x	y	z	f
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

= (above expression)

Min-term / Maxterm expansions

x	y	f
1	1	
1	0	
0	1	
0	0	

each are 0 or 1

Consider $\boxed{x \cdot y}$

x	y	$x \cdot y$	$x \cdot \bar{y}$	$\bar{x} \cdot y$	$\bar{x} \cdot \bar{y}$
1	1	1	0	0	0
1	0	0	1	0	0
0	1	0	0	1	0
0	0	0	0	0	1

Consider each row has a minterm that is **its** **one**

x	y	Minterm of this row
1	1	xy
1	0	$x\bar{y}$
0	1	$\bar{x}y$
0	0	$\bar{x}\bar{y}$

Def Minterm is the product of all variables as v or $\bar{v}$

Minterm expression

x	y	z	f	
1	1	1	1	$\rightarrow xyz$
1	1	0	0	
1	0	1	1	$\rightarrow x\bar{y}z$
1	0	0	0	
0	1	1	0	
0	1	0	0	
0	0	1	0	
0	0	0	1	$\rightarrow \bar{x}\bar{y}\bar{z}$

$$\boxed{f = xyz + x\bar{y}z + \bar{x}\bar{y}\bar{z}}$$