Math 511

Solving Systems of Linear Eqn's

① Substitution

② Elimination → equiv. systems : Same soln set

$$\begin{cases} x + y + z = 3 \\ 2x - y - z = 0 \\ x + 3y - z = 3 \end{cases} \xrightarrow{\text{do "work"}} \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$$

Keeps system equivalent

Soln set $(1, 1, 1)$

Same soln set

equivalent systems ① interchange eqn's
② mult. eqn by a non-zero.
③ eqn + mult. · eqn = new eqn

Matrix — [rect. array of real numbers]

sys of eqn's → $\left[\begin{array}{c|c} \text{coeff matrix} & \text{const matrix} \end{array}\right]$

eqn ops → $\boxed{\text{row ops}}$

① interchange rows
② mult row by non-zero
③ row + mult · row = new row

goal of elimination is to back solve

back solve: $\left\{\begin{array}{l}\text{starting} \\ \text{system}\end{array}\right. \xrightarrow{\text{equiv.}} \text{systems} \left\{\begin{array}{l} x + y + z = 1 \\ 0 \quad 3y + z = 2 \\ 0 \quad 0 \quad 3z = 3 \end{array}\right.$

$z = 1$

ex

$\left\{\begin{array}{l} x + y + 1 = 1 \\ 3y - 1 = 2 \end{array}\right. \rightarrow \left\{\begin{array}{l} x + 1 + 1 = 1 \\ \boxed{x = -1} \end{array}\right.$

$y = 1$

Soln $\boxed{(-1, 1, 1)}$

1.2 get a system into $\boxed{\text{row echelon form}}$ to backsolve?

Def
1. first non-zero row entry is a 1.
2. number of leading zeros of one row is more than row above it. (unless you have all zeros)
3. all zero rows are at the bottom.

Goal system of linear eqn's $\rightarrow$ [coeff | const] $\xrightarrow[\text{row ops}]{\text{do}}$ [row echelon form | const]

back solve this to get sol'n's

this is called $\boxed{\text{Gaussian Elimination}}$

$$\begin{cases} x+y+z=3 \\ 2x-y-z=0 \\ x+3y-z=3 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & -2 & -2 & -3 \\ 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 3 \end{array}\right]$$

$r_1 - r_2 = \cancel{N} r_1$

$$\tfrac{1}{2} r_1 = Nr_1 \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 3 \end{array}\right] \text{ vs } \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & -1 & 0 \\ 1 & 3 & -1 & 3 \end{array}\right]$$

lets use

lead var.

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{1} & -2 & -2 & -3 \\ 0 & 3 & 3 & 6 \\ 0 & 5 & 1 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 5 & 1 & 6 \end{array}\right]$$

$r_2 - r_1 = Nr_2$

$r_3 - r_1 = Nr_3$

$1/3\, r_2 = Nr_2$

$$-5r_2 + r_3 = Nr_3 \left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

by back-solve

$z = 1$

$y = 1 \rightarrow \boxed{(1,1,1)}$

$x = 1$

or Gauss - Jordan

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \rightarrow \begin{matrix} r_2 - r_3 = Nr_2 \\ r_1 + 2r_3 = Nr_1 \end{matrix} \quad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$r_1 + 2r_2 = Nr_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solve → 3 possible ans. ① zero soln's (inconsistent)

② one soln } (consistent)

③ ∞ soln's

System of eqn's → [coef | const] → gaussian elimination

① → look at 1st aug. matrix

a) more rows than col's. (of coef matrix)

then this is overdetermined : guess zero soln's

b) more col's than rows (of coef. matrix

know: ∞ solns.

c) col's = rows

guess : one soln

$\{ x + y = 1$

$$\left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \end{array}\right]$$

lead free

② Consider steps of gaussian elimination.

a)

$$\begin{array}{cccc|c} x & y & z & v & \\ \boxed{1} & 2 & 3 & 4 & -1 \\ 0 & 0 & \boxed{1} & 2 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$v = n_2$

$z = \pi - 2n_2$

$y = n_1$

$x = -1 - 4n_2 - 2n_1 - 3(\pi - 2n_2)$

x, z are lead var's

y, v are free

any free → ∞ soln's

b) $\xrightarrow[\text{elim}]{\text{gauss}} \left[\begin{array}{cccc|c} - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \to 0 = 1$ ¡ no soln.

ex $\begin{cases} x + y = 1 \\ x + y = \pi \end{cases} \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & \pi \end{array}\right]$ $r_1 - r_2 = N r_2$

$\to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1-\pi \end{array}\right]$ no soln.

③ $\left[\begin{array}{ccc|c} 1 & & 0 & \text{const} \\ & 1 & & \\ 0 & & 1 & \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x + y = 1$
$y = 2$
$x + y = 1$
$y = 2$

$\to \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Note: A very special system to know

ex $\begin{cases} x + y + z = 0 \\ 3x - y = 0 \\ 2x + z = 0 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$

any system where every eqn is (linear expression) = 0.

Call system a $\boxed{\text{homogeneous system}}$

Idea of why homogeneous systems being important?

ex college algebra: $3x^3 + x(x^2 - 7) = 2x + 2$

$4x^3 + 0 \cdot x^2 - 9x - 2 = 0$

$4(x+\ )(x+\ )(x+\ ) = 0$

① All homogeneous systems are consistent.

Have at least the all zero trivial soln.

② Question: does it have any non-trivial soln's?

Yes.

$\boxed{\text{If}}$ as you do gauss-elim. you get free variables.