Math 511

Q's 1.2 (6b)

$$\left[\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 3 & 1 & 2 & -1 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & -8 & -1 & -4 & -10 \end{array}\right]$$

$r_2 - r_3 = N r_2$

$- r_3 = N r_3$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & 4 & 12 \\ 0 & 8 & 1 & 4 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 1 & 3 \\ 0 & \boxed{1} & 1/8 & 1/2 & 5/4 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

Swap r_2, r_3

$1/4\, r_3 = N r_3$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1/8 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 5/8 & 0 & 3/4 \\ 0 & 1 & 1/8 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$x_4 = 3$

free $x_3 = \alpha$

$x_2 = -\frac{1}{8}\alpha - 1/4$

$x_1 = -5/8\,\alpha + 3/4$

$x_2 + 1/8\,\alpha = -1/4$

$x_1 + 5/8\,\alpha = 3/4$

6d)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

free $x_4 = \alpha$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array}\right]$$

$x_4 = \alpha$

$x_3 = \frac{1}{3}\alpha$

$x_2 = 0$

$x = -4/3\,\alpha$

$\boxed{1.3}$ Sys. of Linear Equ's $\rightarrow$ Augmented Matrices $\left[\text{coeff} \mid \text{const}\right]$

New toy = Matrix

Notation: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = [a_{ij}]$

Size: $m \times n$

row col

matrix

Special matrices: $1 \times n$ or $m \times 1$ call them vectors (tuples)

n-tuple m-tuple

row vector column vector

ex $[1 \ 2 \ 3]$ $\begin{bmatrix} ? \\ -1 \\ 0 \end{bmatrix}$

Symbols: - bold font for vector
- double side font

$\mathbf{v}$ or $\mathbb{V}$ col. vector

$\underline{\mathbf{v}}$ or $\underline{\mathbb{V}}$ row vector

back to matrix $\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = [a_{ij}]$

$A = [\mathbb{a}_1 \ \mathbb{a}_2 \ \cdots \ \mathbb{a}_n] = \begin{bmatrix} \vec{\mathbb{a}}_1 \\ \vec{\mathbb{a}}_2 \\ \vdots \\ \vec{\mathbb{a}}_m \end{bmatrix}$

toys: Matrices, Vectors

Rules: ① Equality $A = [a_{ij}]$ $B = [b_{ij}]$

$A = B$ if both are $m \times n$ (Same size)

and for ij $a_{ij} = b_{ij}$

Same number in Same place

② **Operations**

a) $A + B = [a_{ij} + b_{ij}]$

both $m \times n$

b) $\alpha \cdot A = [\alpha a_{ij}]$ Scalar multiplication

c) Matrix • matrix Matrix multiplication

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -2 \\ 6 & 2 & 2 \end{bmatrix}$$

2×2 2×3 2×3

why?

$2x - y = 4$ Vars? x, y

1×2

coeff $= [2 \ -1]$ Const $= [4]$

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

2×2

$2x - y = 4$

$x + 2y = 3$ $\rightarrow$ $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Vars? x, y

$$\underset{\text{matrix}}{A}\ \underset{\text{col. vector}}{\mathbb{x}} = \underset{\text{col. vector}}{\mathbb{b}}$$

So $A\mathbb{x} = \mathbb{b}$

but

$$\begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix} \mathbb{x} = \mathbb{b} \quad \text{so} \quad \begin{bmatrix} \vec{a_1}\,\mathbb{x} \\ \vec{a_2}\,\mathbb{x} \\ \vdots \\ \vec{a_m}\,\mathbb{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

So we can consider system of linear eqn's

is $\underset{\text{coeff}}{A}\ \underset{\text{var's}}{\mathbb{x}} = \underset{\text{const.}}{\mathbb{b}}$

solved by

$$\begin{bmatrix} \vec{a_1}\,\mathbb{x} \\ \vec{a_2}\,\mathbb{x} \\ \vdots \\ \vec{a_m}\,\mathbb{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Note: what does $\vec{a_1}\,\mathbb{x} = [a_{11}\ a_{12} \cdots a_{1n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

So

$$A\mathbb{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

So $A = [a_{ij}] = [a_1\ a_2 \dots a_n] = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

$$A x = \begin{bmatrix} \vec{a}_1 x \\ \vec{a}_2 x \\ \vdots \\ \vec{a}_3 x \end{bmatrix} = \underbrace{x_1 a_1 + x_2 a_2 + \dots + x_n a_n}_{\text{linear combination of A's cols.}}$$

$\boxed{\text{Thm}}$ $Ax = b$ has a solution if and only if

b = linear combination of A's columns.

$\boxed{\text{Now}}$ Matrix . Matrix $\underset{m\times n}{A} \cdot \underset{n\times k}{B} = \underset{m\times k}{C}$

$$[\vec{a}_i][b_j] = [c_{ij}]$$

$$c_{ij} = \vec{a}_i \cdot b_j$$
