Math 511

Matrix Arithmetic

① $A + B = [a_{ij} + b_{ij}]$

② $\alpha A = [\alpha a_{ij}]$

③ Matrix Multiplication

$$\underline{A\mathbf{v}} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \mathbf{v} = \begin{bmatrix} \vec{a}_1 \mathbf{v} \\ \vec{a}_2 \mathbf{v} \\ \vdots \\ \vec{a}_m \mathbf{v} \end{bmatrix}$$

$$\underline{A\mathbf{v}} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n$$

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_n]$$

$$AB = C = [c_{ij}] \qquad c_{ij} = \vec{a}_i\, \mathbf{b}_j$$

Note: two special Matrices ..

① $O = \begin{bmatrix} \text{all zeros} \end{bmatrix}$

② $I = [\delta_{ij}]$ $\qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$I = \begin{bmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

typically I is $\underline{n \times n}$ $\qquad I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

1.4 **Matrix Algebras**

objects: A, $\vec{v}$, α

ops: $A+B$, αA, $A \cdot B$, A^T

Laws Th^n

① $A+B = B+A$

② $(A+B)+C = A+(B+C)$

③ $(AB)C = A(BC)$

④ $A(B+C) = AB+AC$

⑤ $(B+C)A = BA+CA$

⑥ $(\alpha\beta)A = \alpha(\beta A)$

⑦ $\alpha(AB) = (\alpha A)B = A(\alpha B)$

⑧ $(\alpha+\beta)A = \alpha A + \beta A$

⑨ $\alpha(A+B) = \alpha A + \alpha B$

⑩ $(A^T)^T = A$

⑪ $(\alpha A)^T = \alpha A^T$

⑫ $(A+B)^T = A^T + B^T$

⑬ $(AB)^T = B^T A^T$

Proof ← show a statement is true.

ex $(A+B)+C = A+(B+C)$

$$(A+B)+C = \left(\left[\left(\underline{a_{ij}+b_{ij}}\right)\right]\right) + C$$

$$= \left[(a_{ij}+b_{ij}) + c_{ij}\right] = \left[a_{ij} + \left(\underline{b_{ij}+c_{ij}}\right)\right]$$

$$= A+(B+C)$$

(vs) example ("verify")

$$\left(\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix} + \begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix}\right) + \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix} = \begin{bmatrix}4 & 2\\ 0 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix} = \begin{bmatrix}6 & 3\\ 1 & 4\end{bmatrix}$$

(vs) $$\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix} + \left(\begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix} + \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix} + \begin{bmatrix}5 & 1\\ 2 & 4\end{bmatrix} = \begin{bmatrix}6 & 3\\ 1 & 4\end{bmatrix}$$

Also b/c $(AB)C = A(BC)$ (assoc. law)

$$A^k = \underbrace{AA\ldots A}_{k\text{-times}}$$

Identity / Inverse of a operation

(ex) **Function Composition:**

Identity is a "do nothing"

$$f(I(x)) = f(x)$$

so $I(x) = x$

Inverse: $? \circ (\sin(x)) = x$

$\sin^{-1}(\ )$

$\arcsin(\ )$

Matrix Addition: $A + \underset{}{\overset{\text{additive identity}}{O}} = A$

Inverse: $A + (-1)A = O$

Matrix Multiplication
A is nxn

$A \cdot I = A \qquad I \cdot A = A$

Inverse: $A \cdot ? = I$

$? \cdot A = I$

Check: $A \cdot B = I \qquad B \cdot A = I$

Call B to be A's inverse $\quad B = A^{-1}$

Note: we will get to finding
① Does A **have** an inverse?

idea: (College Algebra $\quad 3 \cdot X$ can be undone
$\frac{1}{3} \cdot 3 \cdot X = 1 \cdot X = X$

but $\quad$ b/c $\quad 0 \cdot X = 0$
zero does **not** have an inv.)

Matrix Algebra: $A \cdot B$

idea: is $A \cdot B \overset{?}{=} O$

we will find some non-all zero matrices such that

$$A \neq 0 \quad B \neq 0 \quad \text{but} \quad AB = 0$$

it would mean A has no inverse

$\boxed{\text{Def}}$ A is singular or non-invertable if it doesn't have an inverse.

A is not singular or invertable if it does have an inverse. (A^{-1} exists)

For now we only have .. check

$$AB \stackrel{?}{=} I \qquad BA \stackrel{?}{=} I$$

if yes $\Rightarrow$ $B = A^{-1}$
