Math 511

Q's 1.4 #1a

A is non-singular $\equiv$ A^{-1} exists $\equiv$ A is invertable
A is singular $\equiv$ A^{-1} does not exist $\equiv$ A is not invertable

#1a $A_{n \times n}$ if $A^2 = O$, then $I - A$ is nonsingular and $(I-A)^{-1} = I + A$

Note Proofs – show something to true

typically implication is: if $\square$, then $\triangle$

tech: #1 direct: assume $\square$ is true, show $\triangle$ is T.

tech #2 Contrapositive: assume not $\triangle$, then show not $\square$

tech #3 Contradiction: assume $\square$ and not $\triangle$
then show always false.

#1a if $A^2 = O$, show $(I - A)$ is non-singular (means it has an inv.)
assume
and it's inv. is $(I + A)$

Know check: $A \cdot B = I$ $BA = I$ so A, B are inverses.

So multiply: $(I-A)(I+A) = (I-A)I + (I-A)A$
$= (I-A) + IA - AA$
$= I - A + A - A^2 = I + O + O = I$

#14 $A, B_{n \times n}$ if $AB = A$ and $B \neq I$ then A is singular

if ($AB = A$ and $B \neq I$), then (A is singular)

A^{-1} does not exist

Pf: Contradiction: $AB = A$ and $B \neq I$ and A^{-1} exists show always false

so $AB = A$

$A^{-1}AB = A^{-1}A$

$IB = I$

$B = I$

always false.

1.5: system of eqn's $\longleftrightarrow$ Matrix: $A\mathbb{X} = \mathbb{b}$

Gauss Jordan — row ops

$x_1 = a_1$

$x_2 = a_2$

$\vdots$

$x_n = a_n$

$$\left[\begin{array}{ccc|c} 1 & & 0 & a_1 \\ & \ddots & & a_2 \\ 0 & & 1 & a_n \end{array}\right]$$

Matrix Algebra

$$\mathbb{X} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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(1) we will find matrizes E_{type1}, E_{type2}, E_{type3} Elementary Matrizes

based on row ops: row swap, α(row), row + m·row

(2) know $A\mathbb{X} = \mathbb{b}$ has a soln then for M non-singular

$MA\mathbb{X} = M\mathbb{b}$ is equivalent

$\boxed{\text{type 1}}$ (Row Swap)   ex $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$

$E_{type\,1} = \left(I \text{ swap row } i \text{ and row } j \right)$

So $E_{type\,1} M = M$ with row i, row j swapped.

ex $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$

Does $E_{type\,1}^{-1}$ exist?   Yes: b/c $E_{type\,1} \cdot E_{type\,1} = I$

So $E_{type\,1}^{-1} = E_{type\,1}$

$\boxed{\text{type 2}}$ α row i = New row i

$E_{type\,2} = \left(I \text{ with } \alpha \text{ in } \delta_{ii} \text{ position} \right)$

ex $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 7 \end{bmatrix}$ make 1?

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix}$

So Does $E_{type\,2}^{-1}$ exist?   Yes...

$E_{type\,2}^{-1}$ is $\left(\frac{1}{\alpha} \text{ in } \delta_{ii} \text{ position} \right)$

type 3 row i + $M \cdot$ row j = new row i

E_{type3} = (I with M in a_{ij} spot)

ex

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & \boxed{-2/3} & 7 \end{bmatrix}$$

$\underline{\text{row } 3 + \boxed{\tfrac{2}{3}} \text{ row } 2}$ = New row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & -2/3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

Does E_{type3}^{-1} exist? $\underline{\text{Yes}}$

E_{type3}^{-1} = (put $-M$ in a_{ij} position of I)

So $\boxed{\text{th}^{m}}$ if E is elem. then E is non-singular

and $\underline{E^{-1}}$ is of same type.

$\boxed{\text{ex}}$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$ swap row 3, row 1 is $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$\quad\quad \shortparallel$ E_{type1}

and $E_{type1}^{-1} = E_{type1}$

$\boxed{\text{ex}}$

$$\begin{bmatrix} 7 & 1/3 & 4 \\ 2 & 1 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$ $\frac{1}{5}$ row 3 is $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 7 & 1/3 & 4 \\ 2 & 1 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$E_{type2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 1/3 & 4 \end{bmatrix}$$

$row_1 + (-2)\, row_2 = \text{New } row_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 1/3 & 4 \end{bmatrix}$$

$$E^{-1}_{type\,3} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider: System of Eqn's $\longleftrightarrow$ $A\mathbb{x} = \mathbb{b}$

gauss jordan

Step 1	row op	$\longrightarrow$	$E_1 A\mathbb{x} = E_1 \mathbb{b}$
Step 2	row op	$\longrightarrow$	$E_2 E_1 A\mathbb{x} = E_2 E_1 \mathbb{b}$
	$\vdots$		
goal	$\left[\begin{array}{c\|c} I & \text{ans} \end{array}\right]$	$\longrightarrow$	$\boxed{\underline{E_k \cdots E_2 E_1 A}}\,\mathbb{x} = \underline{E_k \cdots E_2 E_1 \mathbb{b}}$

terms: If $B = E_n \cdots E_2 E_1 \cdot A$ call A, B row equivalent

Thm: A is $n \times n$ then the following are logically equiv.

1. A^{-1} exists (A is non-singular)
2. $A\mathbb{x} = \mathbb{0}$ has only the trivial Soln.

   zero vector

3. A is row equiv. to I.

What does this th^{m} give us?

1st we can find A^{-1}

$$A\mathbb{x} = \mathbb{b}$$

$$E_1 A\mathbb{x} = E_1 \mathbb{b}$$

$$\vdots$$

$$\underbrace{\boxed{E_k \cdots E_2 E_1 A}}_{=\ I}\mathbb{x} = E_k \cdots E_2 E_1 \mathbb{b}$$

so $A^{-1} = (\underbrace{E_k \cdots E_2 E_1}_{\text{these are the gauss-jord. steps.}})$

these are the gauss-jord. steps.

so use gauss-jordan:

$$[A \mid I] \qquad E_1 A \qquad E_1 I$$

row ops

$$[I \mid A^{-1}]$$

2nd track ops

gauss

except ignore the need for 1's as lead var.

$$A\mathbb{x} = \mathbb{b}$$

$$E_1 A \mathbb{x} = E_1 \mathbb{b}$$

$$E_2 E_1 A\mathbb{x} = E_2 E_1 \mathbb{b}$$

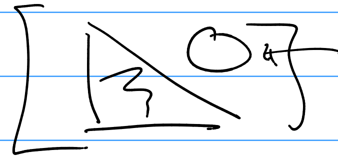
$$\vdots$$

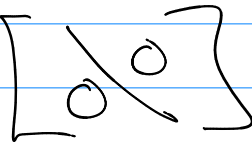
$$\underbrace{\boxed{E_n \cdots E_2 E_1 A}}_{=}\mathbb{x} = E_n \cdots E_2 E_1 \mathbb{b}$$

upper triangular when "done" with gauss part

Def: upper triangular = 

zero's below diagonal

lower triangular =  zero's above

diagonal = 

back to.. $E_n \cdots E_2 E_1 A = U$ (upper triangular)

$$A = \underline{\left(E_1^{-1} E_2^{-1} \cdots E_n^{-1}\right)} U$$

Note: if you restrict yourself to type 3 ops.
and only remove values below diagonal,

then $E_1^{-1}, E_2^{-1}, \ldots, E_n^{-1}$ are **all** lower triang.

and (lower tri)(low tri) = low tri

so $A = (L) U$

lower triangular $= L = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$
upper triangular $= U$
