Math 511

Q's 1.6 Partitioned Matrices

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & \\ \hline & \end{array}\right]$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \\ & \end{pmatrix}$$

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todo: system of eqn's (3x3)

Solve by substitution

Solve by elim. with aug. matrix

row ops $\longleftrightarrow$ $EA\mathbf{x} = E\mathbf{b}$

$A\mathbf{x} = \mathbf{b}$

E and know E^{-1}

Solve $\longleftrightarrow$ $(E_k \cdots E_2 E_1)A\mathbf{x} = E_k \cdots E_2 E_1 \mathbf{b}$

A^{-1}

check: $[A \mid I]$ $\xrightarrow{\text{row ops}}$ $[I \mid A^{-1}]$

check

Soln $x=1$, $y=1$, $z=1$

$$\begin{cases} 2x - y + 3z = 4 \\ 5x + 2y - z = 6 \\ x - 7y - 2z = -8 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 5 & 2 & -1 & 6 \\ 1 & -7 & -2 & -8 \end{array}\right]$$

Note: 1.4 #12 show $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ that $A^{-1} = \boxed{\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$

$AA^{-1} = I$ and $A^{-1}A = I$

B Now to find A^{-1}

$[A \mid I]$ row ops $\to$ $[I \mid A^{-1}]$

fun?

$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \end{array}\right]$

If A is 2x2 it seems from above that

when $a_{11}a_{22} - a_{12}a_{21} = 0$ $\quad A^{-1}$ does not exist

when $a_{11}a_{22} - a_{12}a_{21} \neq 0$ $\quad A^{-1}$ does exist

Def If A is 2x2, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

def $\quad \det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$

A is 1x1 $\quad A = (a_{11})$

$\det(A) = |A| = a_{11}$

$\det(A) = a_{11} = 0 \quad A^{-1}$ does not exist

$\det(A) = a_{11} \neq 0 \quad A^{-1}$ does exist

So far $\boxed{\det\left(\begin{pmatrix} a_{11} \end{pmatrix}\right) = a_{11}}$

$$\det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

what about 3×3? 4×4? .. $n\times n$?

Inductive Rule: Basis Step: $\det\left(\begin{pmatrix} a_{11} \end{pmatrix}\right) = a_{11}$

Inductive Step: ???

3×3

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & \\ 0 & \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23}-a_{21}a_{13}}{a_{11}} & \\ 0 & \frac{a_{11}a_{32}-a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33}-a_{31}a_{13}}{a_{11}} & \end{array}\right]$$

2x2 system

can be solved (A^{-1} exists)

if its det is not zero

also a_{11} must not be zero

it looks like $a_{11} \left| \quad \right| \neq 0$

back to $\det\left((a_{11}) \right) = a_{11}$

$$\det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = a_{11} a_{22} - a_{12} a_{21}$$

Terms Needed

$$\underset{n \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]$$

(1) M_{ij} is the minor or a_{ij}. M_{ij} is all of A except you remove i^{th} row, j^{th} column

ex $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ $\quad M_{21} = \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix}$

(2) $A_{ij} = (-1)^{i+j} |M_{ij}|$ is the cofactor of a_{ij}

ex $A_{21} = \underline{(-1)}^{2+1} \begin{vmatrix} 0 & -1 \\ 1 & 5 \end{vmatrix} = (-1)\left(0 \cdot 5 - (1)(-1)\right)$

$= -1$

Note: $(-1)^{i+j}$ of cofactors $\to$ $\begin{bmatrix} 1 & -1 & 1 & \cdots \\ -1 & 1 & \cdots & \\ 1 & -1 & \cdots & \\ \vdots & \vdots & \vdots & \end{bmatrix}$

$$\det(A) = \begin{cases} a_{11} & \text{if } A = (a_{11}) \\ a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} + \cdots + a_{1n} A_{1n} & \end{cases}$$

row 1 cofactor expansion

i^{th} row $\quad a_{i1} A_{i1} + a_{i2} A_{i2} + a_{i3} A_{i3} + \cdots + a_{in} A_{in}$

j^{th} col $\quad a_{1j} A_{1j} + a_{2j} A_{2j} + a_{3j} A_{3j} + \cdots + a_{nj} A_{nj}$

Ex $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 3 & 1 \\ 1 & 7 & 2 \end{pmatrix}$

$$\det(A) = \cancel{0 \cdot A_{12}} + 3 \cdot A_{22} + 7 \cdot A_{32}$$

$$= +3 \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= 3(5) - 7(6) = 15 - 42$$

$$= \boxed{-27}$$

$$\det \begin{bmatrix} 0 & 3 & -2 & 1 \\ 4 & 0 & 1 & 2 \\ 3 & 0 & -1 & 4 \\ 2 & 1 & 1 & 2 \end{bmatrix} = -3 \begin{vmatrix} 4 & 1 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & -2 & 1 \\ 4 & 1 & 2 \\ 3 & -1 & 4 \end{vmatrix}$$

$$= -3 \left[4 \begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \right]$$

$$+1 \left[2 \begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix} + 1 \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} \right] = ?$$

Properties of det(A)

Th^m $\det(A^T) = \det(A)$

$\boxed{\text{th}^n}$ if A is triangular, $\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \cdots \cdot a_{nn}$

ex

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 1 \begin{vmatrix} 2 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 1 \cdot 2 \begin{vmatrix} 3 & 7 \\ 0 & 4 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot 4$$

$\boxed{\text{th}^n}$

1. if A has an all zero row or all zero col $\longrightarrow |A| = 0$
2. if A has identical rows or columns $\longrightarrow |A| = 0$