Math 511

Q's (a) $\begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \left[\begin{array}{c|c} OB_{11} + IB_{21} & OB_{12} + IB_{22} \\ \hline IB_{11} + OB_{21} & IB_{12} + OB_{22} \end{array}\right]$

$= \begin{bmatrix} B_{21} & B_{22} \\ B_{11} & B_{12} \end{bmatrix} =$

given

$B_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $B_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$B_{21} = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ $B_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1.6 #8

$\boxed{\underset{m\times n}{A} \underset{n\times r}{X} = \underset{m\times r}{B}}$ iff $A\mathbf{x}_j = \mathbf{b}_j$ $j = 1, \dots, r$

X's j's col

B's j^{th} col

$A = [a_{ij}] = [a_1 \; a_2 \cdots a_n] = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

$AX = A[\mathbf{x}_1 | \mathbf{x}_2 | \mathbf{x}_3 | \cdots | \mathbf{x}_r] = [A\mathbf{x}_1 | A\mathbf{x}_2 | \cdots | A\mathbf{x}_r]$

$B = [\mathbf{b}_1 \; \mathbf{b}_2 \; \mathbf{b}_3 \cdots \mathbf{b}_r] = [\mathbf{b}_1 | \mathbf{b}_2 | \cdots | \mathbf{b}_r]$

2.2 / 2.3

det(A) by cofactor expansion

Cost: A is 10x10 → about 3.6 million adds
about 6.2 million mult.

Hope of a faster tech is . .

① $\det(\text{Triangular}) = t_{11} \cdot t_{22} \cdot \ldots \cdot t_{nn} = \prod_{i=1}^{n} t_{ii}$

② A ⟶ row ops . . (gaussian elim)

$E_k \cdots E_2 E_1 A = U$

③ So can we understand $\det(EA) = \begin{bmatrix} ? & \\ 0 & \end{bmatrix}$ det(A) in here?

Type 1 (row swap) $\det(E_{type1} A) = ?$

ex $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

row swap $\det\begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} = a_{21}a_{12} - a_{11}a_{22}$?

for all A. ⟶ $\det(E_{type1} A) = -\det(A)$

try I $\det(E_{type1} I) = -\det(I)$

⟶ $\det(E_{type1}) = -1$

So $\det(E_{type1} A) = \det(E_{type1})\det(A) = (-1)\det(A)$

det of row swapped matrix
= (-1) det of orig. matrix

ex $\det\begin{pmatrix} 0 & 3 & 4 \\ 1 & -1 & 7 \\ 0 & 0 & 2 \end{pmatrix} = ?$ $\boxed{-6}$

swap 1,2 $\det\begin{pmatrix} 1 & -1 & 7 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix} = 6$

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type 2 $\quad \alpha \cdot \text{row}_i = \text{New row}_i \qquad \det(E_{\text{type2}} A) = ?$

$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ vs $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \qquad \alpha a_{21}A_{21} + \alpha a_{22}A_{22} + \alpha a_{23}A_{23}$

$\alpha \det(A)$

so $\det(E_{\text{type2}} A) = \alpha \det(A)$

try I $\quad \det(E_{\text{type2}} I) = \alpha \det(I)$

$\det(E_{\text{type2}}) = \alpha$

so $\det(E_{\text{type2}} A) = \det(E_{\text{type2}}) \det(A) = \alpha \det(A)$

Lemma $\quad$ consider A's i^{th} row $\quad a_{i1}, a_{i2}, a_{i3}, \ldots a_{in}$

consider j^{th} row cofactors $\quad A_{j1}, A_{j2}, A_{j3}, \ldots A_{jn}$

now: $\quad a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$

$$= \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note: collect all cofactors ..

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & & A_{nn} \end{bmatrix}^T$$

type 3 New $\text{row}_i = \text{row}_i + c\,\text{row}_j$

$$E_{type3} = \begin{bmatrix} 1 & & \\ & \ddots & \\ \boxed{c} & & 1 \end{bmatrix} \leftarrow$$

in a_{ij}

$$E_{type3} A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \boxed{\vec{a}_i + c\,\vec{a}_j} \\ a_n \end{bmatrix} \leftarrow i^{th} \text{ row}$$

use this row for cof. expan.

$$\det(E_{type3} A) = (a_{i1} + c\,a_{j1}) A_{i1} + (a_{i2} + c\,a_{j2}) A_{i2} + \cdots$$

$$= \det(A) + 0$$

So, .. $\det(E_{type3} A) = \det(A)$

try I $\quad \det(E_{type3} I) = \det(I)$

$\det(E_{type3}) = \det(I) = 1$

So E of type1, type2, type3

$$\det(EA) = \det(E)\det(A)$$

and

$$\det(E) = \begin{cases} 1 & \text{if type3} \\ -1 & \text{if type 1} \\ \alpha & \text{if type 2} \end{cases}$$

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ -2 & 1 & 0 & 2 \end{pmatrix} = \boxed{3}\ \boxed{\det\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix}}\ \circledast$$

type 3

$$\circledast = \det\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 3(5) = 15$$

$\Rightarrow$ So $\det(A) = \boxed{3}(15) = \boxed{45}$

If you use row ops remember --

$$E_k \cdots E_2 E_1 A = U \leftarrow \text{triangular}$$

$$\boxed{\det(U) = \prod_{i=1}^{n} u_{ii}} \quad \text{easy to find.}$$

So $\det(E_k)\det(E_{k-1})\cdots\det(E_2)\det(E_1)\underline{\det(A)} = \prod_{i=1}^{n} u_{ii}$

$$\boxed{\det(A) = \frac{\prod u_{ii}}{\det(E_k)\cdots\det(E_1)}}$$

A is non-singular iff it is row equiv. to I

A is singular iff is is not row equiv. to I

Th^m A is singular iff $\det(A) = 0$

A is non-singular iff $\det(A) \neq 0$

2.3

$$\mathrm{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & & A_{nn} \end{bmatrix}^T$$

Consider:

$$A \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ & & & \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & 0 & & \\ 0 & & \ddots & & \\ & & & & \det(A) \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)}$$