Math 511

$$\underset{m\times n}{A} = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [\,a_1\; a_2 \cdots a_n\,] = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$$

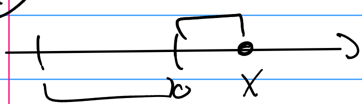
Vector

row Vector

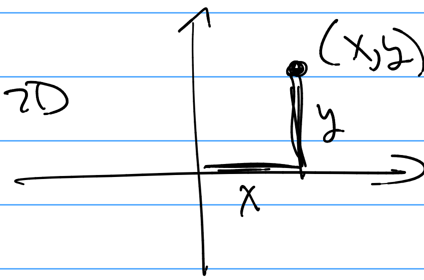
Consider "Space"

ex) 1D

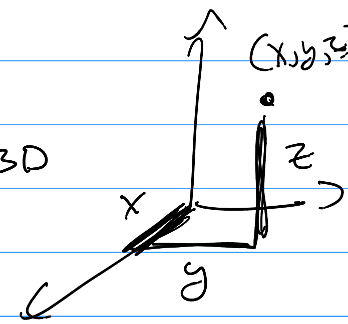
Location



2D

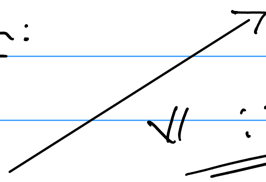


3D



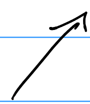
direction and magnitude

any dimension:



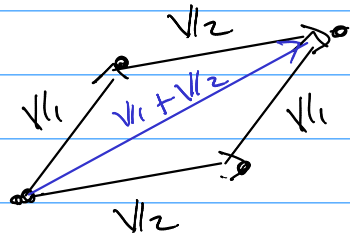
$\mathbb{V}$: Vector

New toy: Vector



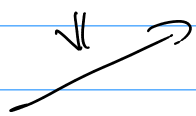
Ops: ① Same? Free of location, same direction and same magnitude

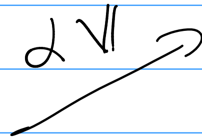
② add?



parallelogram rule

③ Scalar mult.

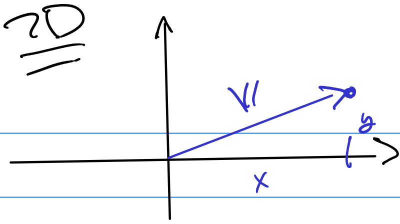




change magnitude by α.
if $\alpha < 0$, flip direction 180°

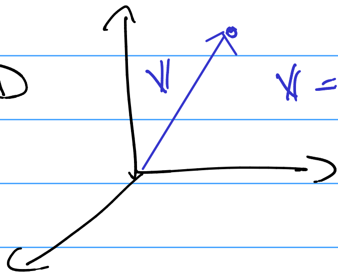
Vectors and numbers

2D



So $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

3D



$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

nD $\quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

① equality $\quad \mathbf{v} = \mathbf{u}$ is $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \overset{\text{equal}}{=} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$

② Sum $\quad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$

③ Scalar mult. $\quad \alpha \mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$

Can we get magnitude? – ex $\quad |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

Can we get direction? $\quad$ (later)

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to get some idea of a "space" a set of objects ("vectors")
and the ability to get around it by --
① stretch $\quad \alpha \mathbf{v}$
② add $\quad \mathbf{v} + \mathbf{u}$

$\boxtimes$ We have a vector space if $\mathbf{v}$ (the objects) and $\alpha \mathbf{v}$, $\mathbf{v} + \mathbf{u}$
satisfy .. 10 axioms

Closure axioms:

c1) $\alpha \mathbf{v}$ is in V

c2) $\mathbf{v} + \mathbf{u}$ is in V

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addition axioms:

A1) $\mathbf{v} + \mathbf{u} = \mathbf{u} + \mathbf{v}$

A2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$

A3) there is a $\mathbf{0}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$

A4) for all $\mathbf{v}$ in V there is a $(-\mathbf{v})$ such that $(\mathbf{v}) + (-\mathbf{v}) = \mathbf{0}$

Scalar mult. axioms

A5) $\alpha(\mathbf{v} + \mathbf{u}) = \alpha \mathbf{v} + \alpha \mathbf{u}$

A6) $(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$

A7) $(\alpha \beta)\mathbf{v} = \alpha(\beta \mathbf{v})$

A8) $1 \cdot \mathbf{v} = \mathbf{v}$

So.. given a set of objects, V, and two operations

$\alpha \mathbf{v}$, $\mathbf{v} + \mathbf{u}$

and all 10 axioms hold $\rightarrow$ call this a **Vector Space**

Obviously $\mathbb{R}^2$ (2D), $\mathbb{R}^3$ (3D), .., $\mathbb{R}^n$ (nD)

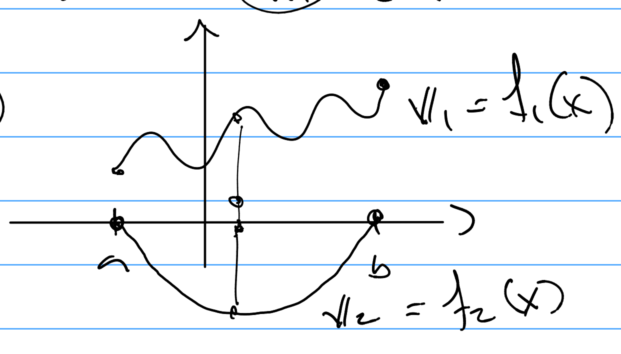
are all vector spaces (check!)

using normal scalar mult

and vector addition.

V is set of all continuous functions over $[a, b]$

ex



$\boxed{\alpha \mathbf{v} \,?}$ def: $\alpha f(x)$

$\boxed{\mathbf{v}_1 + \mathbf{v}_2 \,?}$ def: $f_1(x) + f_2(x)$

is the above a vector space?

check (1) is $\alpha f(x)$ a function? yes

(2) is $f_1(x) + f_2(x)$ a function? yes. $(f_1+f_2)(x)$

A1) $(f_1+f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2+f_1)(x)$

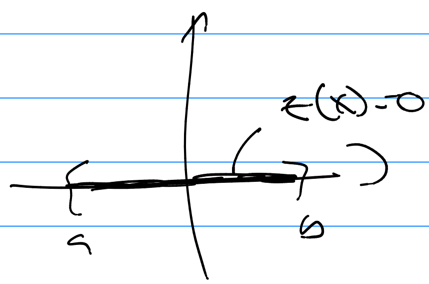
A2) $(f_1+(f_2+f_3))(x) = f_1(x) + (f_2(x) + f_3(x))$

$= (f_1(x) + f_2(x)) + f_3(x) = ((f_1+f_2)+f_3)(x)$

A3) we need a $\boxed{\text{zero function}}$ so that $V + 0 = V$

$z(x) = 0$

yes



$(f+z)(x) = f(x)$

$f(x) + z(x) = f(x)$

$f(x) + 0 = f(x)$

true!

A4) we need add. inverses $\boxed{f(x)} + (\text{its inv.}) = 0$

$$f(x) + \underbrace{((-1)f(x))}_{\text{add inv.}} = 0$$

A5) $\alpha(f_1+f_2)(x)$

$= \alpha(f_1(x) + f_2(x)) = \alpha f_1(x) + \alpha f_2(x)$ ✓

A6) $(\alpha+\beta) f_1(x) = \alpha f_1(x) + \beta f_1(x)$ ✓

A7) $(\alpha\beta) f_1(x) = \alpha(\beta f_1(x))$

A8) $1 \cdot f(x) = f(x)$

So this is a Vector Space, use $\boxed{C[a,b]}$ to represent it

Other spaces: $\mathbb{R}^{m \times n}$

$$\mathbb{V} = \boxed{\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \\ v_{m1} & \cdots & v_{mn} \end{bmatrix}}$$

$\alpha \mathbb{V}$ is normal scalar mult. of matrices

$\mathbb{V}_1 + \mathbb{V}_2$ is normal matrix add.

P_n (n-term polynomial space)

$$\boxed{p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}}$$
