Math 511

Vector Space: set of objects V with $V+U$, αV and **10 axioms**,

called "vectors"

Note: **Induction / Recursion**

1. given set of starting objects (Basis)
2. take "old" object and make new ones (Inductive / Recursive)

ex

Basis: $V = 5$

Inductive Step: if $e_1 \in S$, $e_2 \in S$ then $e_1 + e_2 \in S$

$e_1 - e_2 \in S$

5

$\downarrow$

$5+5 = 10$ $\quad$ $5-5 = 0$ $\quad$ still have 5 $\quad$ $0, 5, 10$

again

$-10, -5, 0, 5, 10, 15, 20$

So $\quad S = \{ \ldots, -15, -10, -5, 0, 5, 10, 15, \ldots \}$

ex

basis: $0, 1$ $\quad$ $a_0 = 0$, $a_1 = 1$

Inductive: $a_n = a_{n-1} + a_{n-2}$

do it $\rightarrow$ $0, 1, 1, 2, 3, 5, 8, \ldots$

ex Basis: $a_0 = 1$

$2^n \quad 2 \cdot 2^{n-1}$

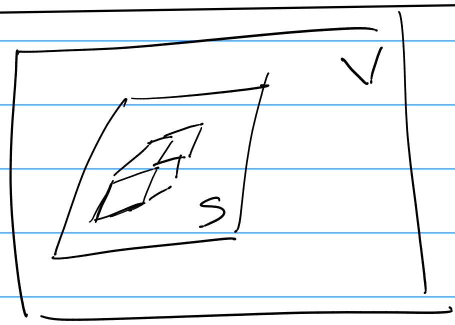
Inductive: $\boxed{a_n} = 2 \cdot \boxed{a_{n-1}}$

guess → $1, 2, 4, 8, 16, \ldots$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

vs closed rule (function) $\boxed{a_n = 2^n}$

3.2



$v + w$, αv

Def S a non-empty subset of V such that

① $x \in S \Rightarrow \alpha x \in S$

② $x \in S, y \in S \Rightarrow x + y \in S$

call S a subspace of V.

two obvious ones ① V is a subspace of V.

② $\{0\}$ is a subspace of V (Zero subspace)

others are called a proper subspace.

How to show S is a subspace of V?

(use the fact S is also a vector space)

① check non-empty, so just check for 0.

② check the two closure properties --

$v + w \in S$, $\alpha v \in S$

ex $V = \mathbb{R}^{2\times 2}$ → any 2x2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

S is $\begin{bmatrix} b & a \\ a & -b \end{bmatrix}$ ex $A = \begin{bmatrix} 5 & 2 \\ 2 & -5 \end{bmatrix} \in S$

$B = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \notin S$

is S a subspace?

check: ① $\vec{0}$ .. is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{\in} S$ yes!

② closure: $\vec{v} + \vec{u}$, $\alpha \vec{v}$

a) $\begin{bmatrix} b & a \\ a & -b \end{bmatrix} + \begin{bmatrix} c & d \\ d & -c \end{bmatrix} = \begin{bmatrix} (b+c) & (a+d) \\ (a+b) & -(b+c) \end{bmatrix} \in S$! true!

b) $\alpha \vec{v} \to \alpha \begin{bmatrix} b & a \\ a & -b \end{bmatrix} = \begin{bmatrix} \alpha b & \alpha a \\ \alpha a & -\alpha b \end{bmatrix} \in S$! true!

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ex is all 3-term polynomials with constant term $= 1$ a subspace of P_3?

anything in P_3 → $a + bx + cx^2 = cx^2 + bx + a$

any S is $1 + bx + cx^2$

check for subspace ..

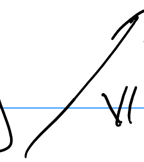
① $\vec{0}$ → $0 + 0x + 0x^2 \notin S$ so not a subspace

② closure

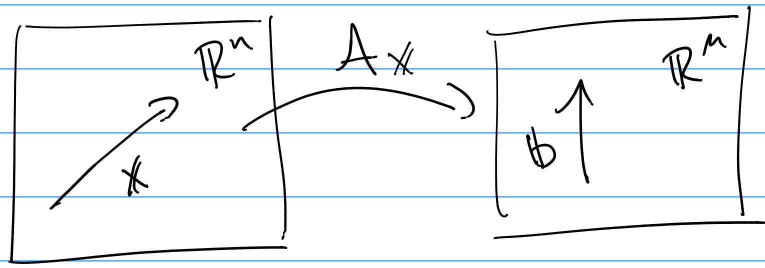
$p + q$ → $(1 + ax + bx^2) + (1 + cx + dx^2)$

$= (2) + (a+c)x + (b+d)x^2 \notin S$

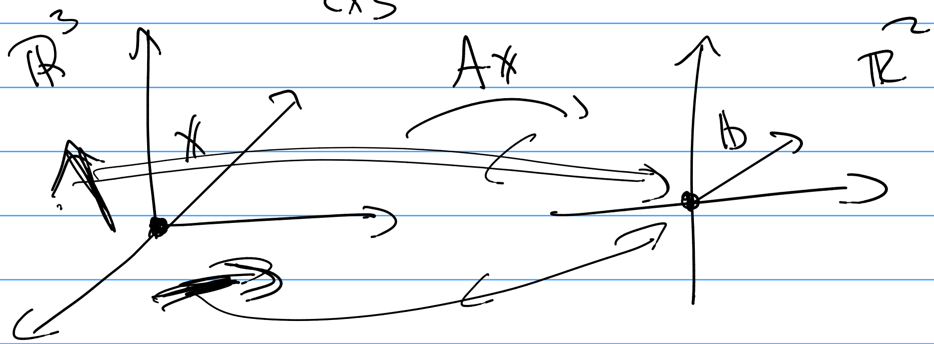
Restrict ourselves to $\mathbb{R}^n$ for a bit... $\mathbb{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$



Consider $\underset{m\times n}{A}\ \underset{n\times 1}{\mathbb{x}} = \underset{m\times 1}{\mathbb{b}}$



ex $\underset{2\times 3}{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}\ \underset{3\times 1}{\mathbb{x}} = \underset{2\times 1}{\mathbb{b}}$



Also: $A\mathbb{x} = \mathbb{0}$ collect all $\mathbb{x}$'s such that $A\mathbb{x} = \mathbb{0}$

Null space — $\boxed{N(A) = \{\mathbb{x} \mid A\mathbb{x} = \mathbb{0}\}}$

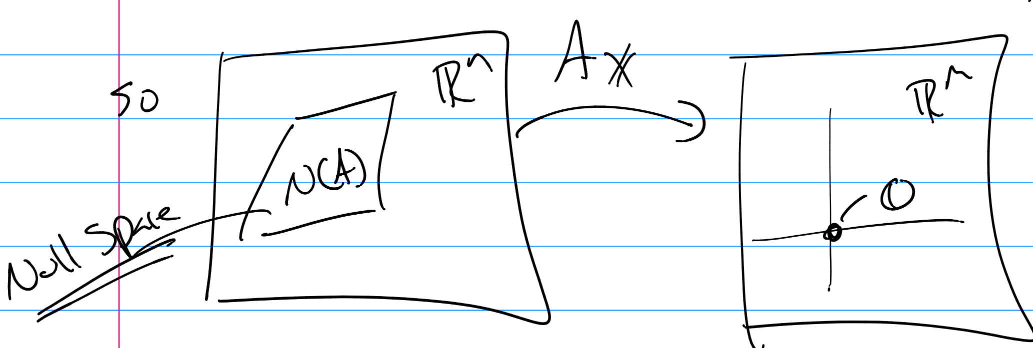
$A\mathbb{x} = 0$ $A\mathbb{y} = 0$ $A(\mathbb{x}+\mathbb{y}) = A\mathbb{x} + A\mathbb{y} = \mathbb{0}+\mathbb{0}$

$= \mathbb{0}$

$A(2\mathbb{x}) = 2A\mathbb{x}$

$= 2\mathbb{0} = \mathbb{0}$

So



back to $Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

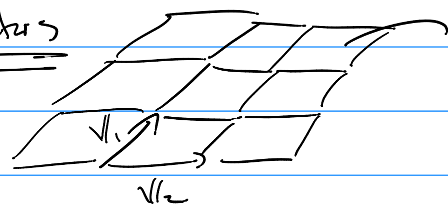
$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$ linear combination of a_i

generalize this idea:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = u$$

given a set of $v_1, v_2, \ldots, v_n$

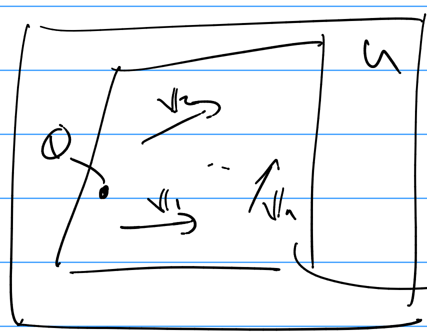
ex) using 2 vectors



all these u's

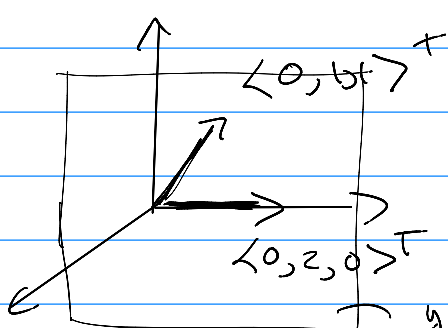
collect all of these $\rightarrow$ $\mathrm{Span}(v_1, v_2, \ldots, v_n)$

$\boxed{\text{Thm}}$ if $v_1, v_2, \ldots, v_n \in V$ (a vector space) then $\mathrm{Span}(v_1, v_2, \ldots, v_n)$ is a subspace of V.



$\mathrm{span}(v_1, v_2, \ldots, v_n)$

ex)

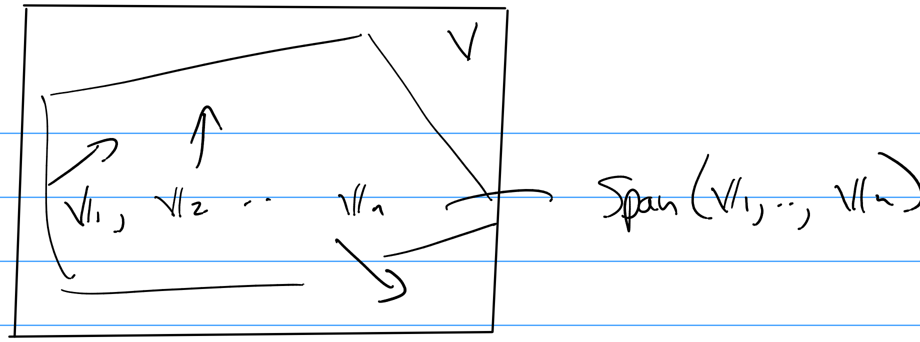


yz plane = Span( )

$\mathrm{span}(v_1, v_2)$ is any

$$\alpha_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_1 + 2\alpha_2 \\ \alpha_1 \end{bmatrix}$$

New Idea



Q: is it possible for span$(v_1, \dots, v_n) = V$

if yes .. call $v_1, v_2, \dots, v_n$ to be a spanning set of V.

Ex $V = \mathbb{R}^3$ ⤳ any where $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

does: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ span $\mathbb{R}^3$?

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

does this have a soln? $\longrightarrow$ $\begin{bmatrix} \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 - \alpha_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{cases} \alpha_1 + 2\alpha_3 = a \\ \alpha_1 + \alpha_2 - \alpha_3 = b \\ 0 = c \end{cases}$$

— no!

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ex $\mathbb{R}^3$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

check: $\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

does $\begin{cases} \alpha_1 + \alpha_3 = a \\ \alpha_1 + 2\alpha_3 = b \\ \alpha_2 + 3\alpha_3 = c \end{cases}$ have a soln.

$\alpha_1 =$

$\alpha_2 =$

$\alpha_3 =$