Math 511

Q's HW → you can put in Math office mailbox on Friday if needed.

3.2 #5 Subspace of P_4?

1st any Vector (polynomial) of P_4 is $V = c_3x^3 + c_2x^2 + c_1x + c_0$

a) S all polynomials in P_4 of even degree

degree of a poly = highest power

even degree is

ex $p \in S$ $\quad p = x^2 = 0x^3 + 1x^2 + 0x + 0$

$p = -x^2 + x = 0x^3 + -1x^2 + 1x + 0$

check for subspace

① is $z = 0x^3 + 0x^2 + 0x + 0x^0$ even degree? yes b/c degree is 0.

② closed?

a) $V + W$ is still in S?

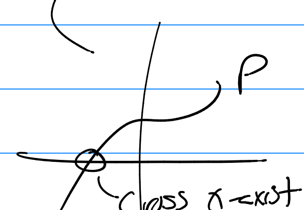
Counter example ex $(-x^2 + x) + (x^2 + x - 3) = 2x - 3$ not even degree

(No, not a subspace)

5d S is set of all poly. in P_4 with at least one real root.

→ to find roots you set equal to zero.

$$\boxed{c_3x^3 + c_2x^2 + c_1x + c_0 = 0}$$



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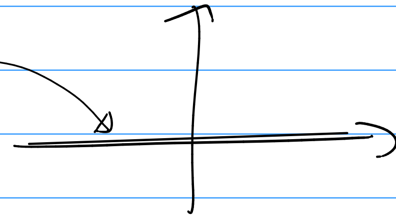
So S is $c_3x^3 + c_2x^2 + c_1x + c_0 = 0$ has at least one soln.

① is $0 = 0x^3 + 0x^2 + 0x + 0$

has infinite solns.



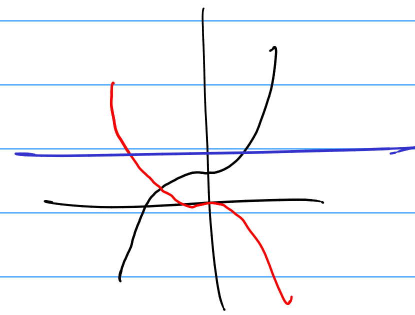
② closure?

counterexample

$x^3 + 1$ has soln

$+ \; -x^3$ has soln

1 no root!



No

3.2 #12e

Spanning set of V: all $v \in V$ can be found by

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = v$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_1 + 2\alpha_2 \\ 3\alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

makes system

$$\boxed{\alpha_1 = a}$$
$$\alpha_1 + 2\alpha_2 = b \rightarrow a + 2\alpha_2 = b \rightarrow \boxed{\alpha_2 = \frac{b-a}{2}}$$
$$3\alpha_1 + \alpha_2 = c \rightarrow 3a + \alpha_2 = c \rightarrow \alpha_2 = \boxed{c - 3a}$$

Solve $\alpha_1 = ?$

$\alpha_2 = ?$

$\alpha_3 = ?$

no soln

Continuing

Goal: to find $\text{Span}(v_1, v_2, \dots, v_n) = V$ (Spanning set)

that is "best".

ex) $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\boxed{\vec{v}_3}\}$

Concepts

3.3 we have no $\boxed{\vec{v}_3} = \boxed{d_1\vec{v}_1 + d_2\vec{v}_2}$

then $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$
$= \mathrm{Span}(\vec{v}_1, \vec{v}_3)$
$= \mathrm{Span}(\vec{v}_2, \vec{v}_3)$

3.4 have smallest number of $\vec{v}_i$

3.5 on any way to compare two spanning sets

3.3 Linear Independence

when does something like

ex) $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ $\vec{v}_3 = \boxed{\vec{v}_1 + 2\vec{v}_2 - \vec{v}_4}$

happen? when does it not?

Consider: $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2 - \vec{v}_4 \rightsquigarrow \boxed{\vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 - \vec{v}_4} = \vec{0}$

$$\rightsquigarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Def: ① $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent (they can not be written as multiples of each other) if

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}}$$

has only trivial soln $c_1 = 0, c_2 = 0, \ldots, c_n = 0$

② $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent (they can be written as multiples of each other) if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

has some non-trivial soln. (one of the $c_i \neq 0$)

So we can check $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ for <u>any</u> V.

ex $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are they linearly ind.?

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for c_1, c_2

$$\begin{bmatrix} c_1 + 3c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$$

$\boxed{\text{ind}}$

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in P_3 $\quad p_1 = 2x - 1 \quad p_2 = x^2 + 1 \quad p_3 = 2x^2 + 2x + 1$

check: $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$c_1(2x-1) + c_2(x^2+1) + c_3(2x^2+2x+1) = 0x^2 + 0x + 0$$

$$(c_2 + 2c_3)x^2 + (2c_1 + 2c_3)x + (-c_1 + c_2 + c_3) = 0x^2 + 0x + 0$$

Solve

x^2: $c_2 + 2c_3 = 0$

x: $2c_1 + 2c_3 = 0$

const: $-c_1 + c_2 + c_3 = 0$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 \quad c_2 \quad c_3$ — free variable → so <u>infinite sol'ns's</u>

$\therefore$ $\boxed{\text{linearly dep}}$

Note: if we restrict ourselves to $V = \mathbb{R}^n$ and consider $X_1, X_2, \dots, X_n$

then $c_1 X_1 + c_2 X_2 + \dots + c_n X_n = 0$

$$\underbrace{\boxed{[X_1 \; X_2 \; \dots \; X_n]}}_{\text{is } n \times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

X " $\qquad$ $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ " C

$\Rightarrow \underset{n \times n}{X} C = 0$

having non-trivial soln's means $X = [X_1 \; X_2 \dots X_n]$ is singular

having only trivial soln means X is non-singular

Th^n $X = [X_1 \; X_2 \dots X_n]$ and $X_i \in \mathbb{R}^n$

a) linearly ind iff X is non-singular $(\det(X) \neq 0)$

b) linearly dep iff X is singular $(\det(X) = 0)$

ex) $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $a_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $a_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$

check! $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$

so $\boxed{\text{linearly dep}}$

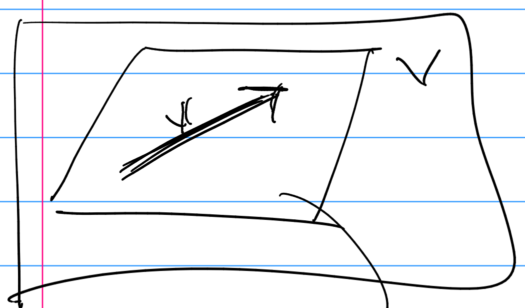
(back to any $\mathbb{V}$)

thm given $\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_n$ in $\mathbb{V}$

$\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_n$ are linearly ind. if and only if

for any $\mathbb{V} \in \mathrm{Span}(\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_n)$

can be uniquely written as $\mathbb{V} = \alpha_1 \mathbb{V}_1 + \alpha_2 \mathbb{V}_2 + \dots + \alpha_n \mathbb{V}_n$



b/c uniq. $\alpha_1, \alpha_2, \dots, \alpha_n$ can be considered as coordinates

span($\mathbb{V}_1, \dots \mathbb{V}_n$) ← is a subspace

Now: $C_{[a,b]}$

→ $\boxed{c_1} f_1''(x) + \boxed{c_2} f_2''(x) + \dots + \boxed{c_n} f_n''(x) = 0$

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⋮

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