Math 511

Q's 3.2 #4a

$$N(A) = \{ \mathbf{x} \mid \underbrace{\boxed{A\mathbf{x} = 0}}_{}\}$$

Solve

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3×4 4×1 3×1

Solve:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free free

$x_2 = \alpha$

$x_4 = \beta$

row 2 $x_3 + 3x_4 = 0$

$x_3 = -3\beta$

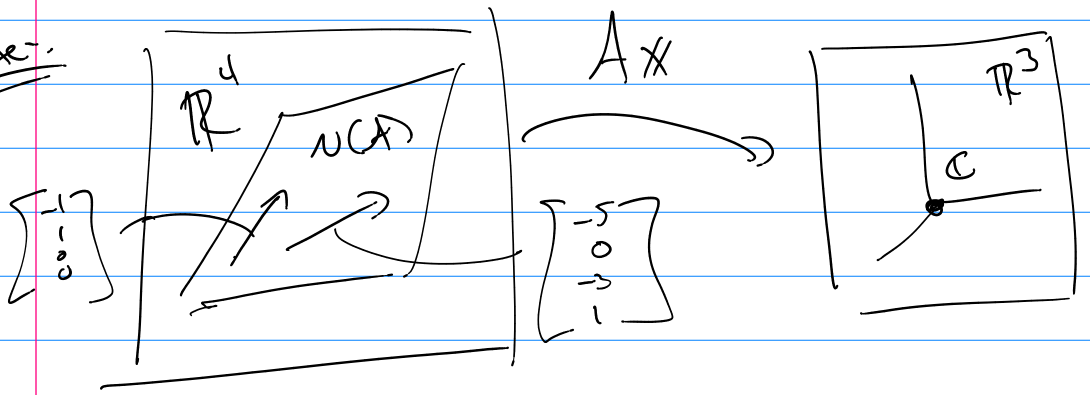
row 1 $x_1 + x_2 - x_3 + 2x_4 = 0$

$x_1 + \alpha + 3\beta + 2\beta = 0$

$x_1 = -\alpha - 5\beta$

$$\mathbf{x} = \begin{bmatrix} -\alpha - 5\beta \\ \alpha + 0\beta \\ 0\alpha - 3\beta \\ 0\alpha + \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Note:



Reminder $V_1, V_2, \dots, V_n$ in V

to check lin. ind. $\quad c_1 V_1 + c_2 V_2 + \dots + c_n V_n = 0$

① only trivial soln $c_i = 0$ for all i → linearly ind.

② have some non-trivial soln → linearly dep.

Ex **Diff Eq:** $f + f'' = 0$

$f = \sin(x) \qquad f = \cos(x)$

$C^{(n-1)}[a,b]$ lin. ind?.

$$\begin{aligned} c_1 f_1 + c_2 f_2 + \dots + c_n f_n &= 0 \\ c_1 f_1' + c_2 f_2' + \dots + c_n f_n' &= 0 \\ c_2 f_1'' + c_2 f_2'' + \dots + c_n f_n'' &= 0 \\ &\vdots \\ c_2 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \dots + c_n f_n^{(n-1)} &= 0 \end{aligned}$$

Ex

$c_1 \sin x + c_2 \cos x = 0$

$c_1 \cos x - c_2 \sin x = 0$

$$\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & \dots & & f_n^{(n-1)} \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

↑ square matrix of reals

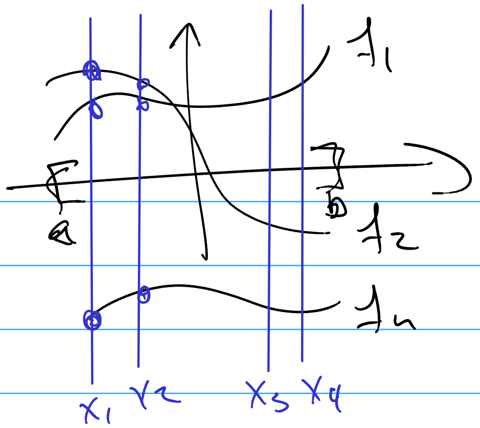
→ Normally use $\det(\ ) \neq 0$ to show linear ind.

$$\det \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & \dots & & f_n^{(n-1)}(x) \end{pmatrix} = W_{f_i}(x)$$

(wronskian)

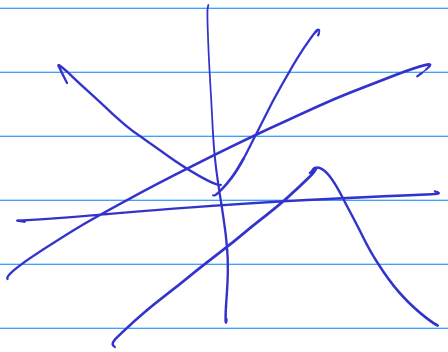
Thm if for any $x \in [a,b]$ $W_{f_i}(x) \neq 0$ → lin. ind. of functions another tech. $f_1, f_2, \dots, f_n$

$$c_1f_1 + c_2f_2 + \dots + c_nf_n = 0$$



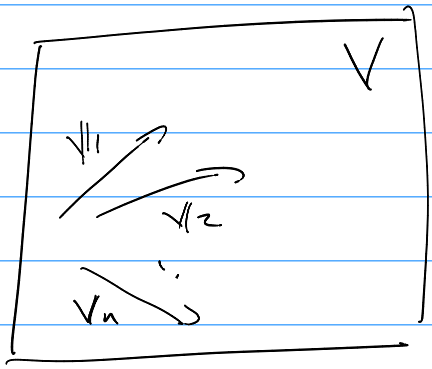
x_1 $\quad c_1\boxed{f_1(x_1)} + c_2\boxed{f_2(x_1)} + \dots + c_n\boxed{f_n(x_1)} = 0$

x_2 $\quad c_1f_1(x_2) + c_2f_2(x_2) + \dots + c_nf_n(x_2) = 0$



3.4 Basis / Dimension

Set $\{v_1, v_2, \dots, v_n\}$



and you find ① $v_1, v_2, \dots, v_n$ are lin. ind.

② $\text{Span}(v_1, v_2, \dots, v_n) = V$ (spanning set)

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \boxed{}$$

"everywhere" vector.

P_3 : $a + bx + cx^2$

$\mathbb{R}^4$: $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

(ex) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

a) linearly ind? . $\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = (-1)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

so $\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0$ so yes lin. ind.

b) Spanning set?

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{aligned} \alpha_1 + \alpha_3 &= a \\ \alpha_1 &= b \\ \alpha_2 + \underset{a-b}{(\alpha_3)} &= c \end{aligned} \quad \longrightarrow \quad \begin{aligned} \alpha_3 &= a-b \\ \alpha_1 &= b \\ \alpha_2 &= c+b-a \end{aligned}$$

Thm if $\{v_1, .., v_n\}$ is a spanning set of V, then any collection of $m > n$ vectors in V are lin. dependent.

Corollary if both $\{v_1, .., v_n\}$ and $\{u_1, .., u_m\}$ are lin. ind. and span V.
then: $n = m$

So the uniq. property of spanning sets that are lin. ind.
→ they always have same number of vectors.

Def: a $\boxed{\text{basis set}}$ $\{v_1, v_2, .., v_n\}$ of V are lin. ind. and they span V.

Def: b/c number of vectors in any basis set of V is uniq. call that the dimension of V.

Ex b/c $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are $\boxed{a}$ basis of $\mathbb{R}^3$

→ dimension of $\mathbb{R}^3$ is 3.

Note: dim of $\{0\}$ is 0

Note: if the basis set is finite then dimension of V is called finite.

ⓔⓧ P is the set of (all) polynomials.

guess dimension of P is finite, call it $\boxed{n.}$
by thm any more than 3 must be linearly dep.

So.. consider

check ind. by wronskien

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 3\cdot 2x & \cdots & n(n-1)x^{n-2} \\ 0 & 0 & 0 & 3! & \cdots & n(n-1)x^{n-3} \\ \vdots & \vdots & & & \ddots & \\ & & & & & n! \end{vmatrix} \neq 0$$

($n+1$ polys)

So P is not finite dimensional..

call such V infinite dimensional

Thm V of dimension $n > 0$, then

① any set of n vectors that are linearly ind, span V and form a basis of V.

② any spanning set of n vectors are lin. ind and form a basis of V.

Thm dimension of V is $n > 0$

① no set of m vectors with $m < n$ span V.

(<u>also</u> ② if $m > n$ → lin. dep.)