Math 511

Q's 3.3 #7 X_1, X_2, X_3 are linear. ind.

$Y_1 = X_1 - X_2$

$Y_2 = X_2 - X_3$ are Y_i lin. ind.

$Y_3 = X_3 - X_1$

linear Ind?: $V_1, V_2 \dots V_n$ in V

Solve $c_1 V_1 + c_2 V_2 + \dots + c_n V_n = 0$ for c_i

only trivial soln $\Rightarrow$ ind.

other soln $\Rightarrow$ dep.

check: $c_1 Y_1 + c_2 Y_2 + c_3 Y_3 = 0$

$c_1(X_1 - X_2) + c_2(X_2 - X_3) + c_3(X_3 - X_1) = 0$

$(c_1 - c_3) X_1 + (c_2 - c_1) X_2 + (c_3 - c_2) X_3 = 0$

b/c X_i are lin ind.

just solve this

$$\begin{cases} c_1 - c_3 = 0 & c_1 = c_3 \\ c_2 - c_1 = 0 & \\ c_3 - c_2 = 0 & \end{cases} \rightarrow \begin{array}{ll} c_2 - c_3 = 0 & \rightarrow c_2 = c_3 \\ c_3 - c_2 = 0 & \rightarrow c_2 = c_3 \end{array}$$

c_3 is free. let it be $1 = c_3 \rightarrow c_1 = 1 \quad c_2 = 1$

non-trivial soln so $\boxed{\text{dep}}$

3.3 (9d) $V = C[0,1]$ $\quad v_1 = f_1(x) = e^x$

$v_2 = f_2(x) = e^{-x}$

$v_3 = f_3(x) = e^{2x}$

$$\begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} \underset{\substack{r_1 - r_2 = Nr_2 \\ r_1 - r_3 = Nr_3}}{=} \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ 0 & 2e^{-x} & -e^{2x} \\ 0 & 0 & -3e^{2x} \end{vmatrix} = (e^x)(2e^{-x})(-3e^{2x})$$

$$= -6e^{2x}$$

Notice @ $x=0$ $\quad W(0) = -6 \neq 0$

So $\boxed{\text{lin. ind.}}$

Basis / Dimension:

linearly ind

$\mathrm{Span}(v_1, \ldots, v_n) = V$

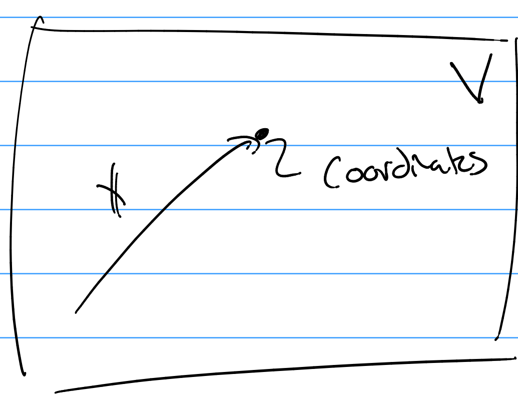
V has a basis of $\boxed{v_1, v_2, \ldots, v_n}$

call dimension of V $\quad \dim(V) = n$

b/c linearly ind. $\quad x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

c_i are uniq.

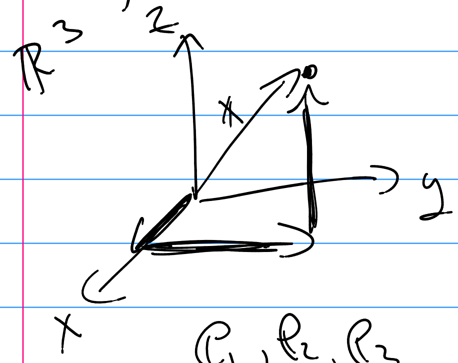
So..



of x in basis $v_1, v_2, \ldots, v_n$

$$[x] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

obviously the standard basis is our "typical" coordinate system.

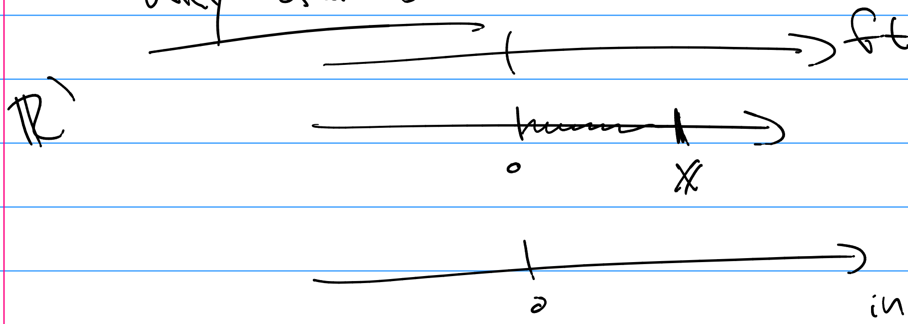


e_1, e_2, e_3

$$\mathbf{x} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

Why other basis?

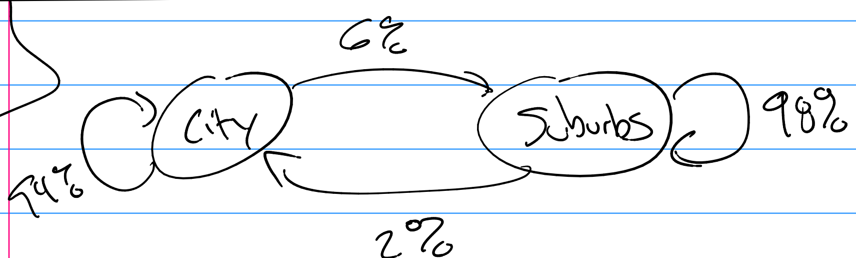


$[2.1]$ ft

$[25.2]$ in

$$\left(\frac{12 \text{ in}}{1 \cancel{\text{ft}}}\right)[2.1]\cancel{\text{ft}} = [25.2] \text{ in}$$

$\boxed{\mathbb{R}^2}$



$$A = \begin{array}{c|cc} & \text{from city} & \text{from sub} \\ \hline \text{to city} & .94 & .02 \\ \text{to sub} & .06 & .98 \end{array}$$

$$\mathbf{x} = \begin{bmatrix}\text{city}\\ \text{sub}\end{bmatrix} \qquad A\mathbf{x}$$

$$\begin{bmatrix}\text{from city to city} & \text{from sub to city}\\ \text{from city to sub} & \text{from sub to sub}\end{bmatrix}\begin{bmatrix}\text{city}\\ \text{sub}\end{bmatrix} = \begin{bmatrix}\text{city}\\ \text{sub}\end{bmatrix}$$

so $AX_0 = X_1$ seq $X_0, X_1, X_2, X_3, \ldots$

$AX_1 = X_2$

$AX_2 = X_3$ Markov chain

$\vdots$

Markov process

$X_1 = AX_0$

$X_2 = AX_1 = A(AX_0) = A^2 X_0$

$\vdots$

$X_n = A^n X_0$

"Study" A and notice $A\begin{bmatrix}1\\3\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\3\end{bmatrix}$

(cont. above example) $A\begin{bmatrix}-1\\1\end{bmatrix} = 0.92\begin{bmatrix}-1\\1\end{bmatrix}$

check: (ind?) $\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$ are they ind?

use $\begin{vmatrix}1 & -1\\3 & 1\end{vmatrix} = 4 \neq 0$ so ind.

So they are a basis.

any vector $X = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}$

$X = d_1\begin{bmatrix}1\\3\end{bmatrix} + d_2\begin{bmatrix}-1\\1\end{bmatrix}$

but $A^n X_0 = A^n\left(d_1\begin{bmatrix}1\\3\end{bmatrix} + d_2\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$= d_1 A^n\begin{bmatrix}1\\3\end{bmatrix} + d_2 A^n\begin{bmatrix}-1\\1\end{bmatrix}$

$$A^n x_0 = d_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d_2 \cancel{(.92)^n} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{change of basis}}$ Vector space of dimension n.

#1 basis $v_1, v_2, \dots, v_n$ let $V = [v_1 \ v_2 \dots v_n]$

by uniq. coord. $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$$x = V[c]_v \qquad [c]_v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$ coord of basis v for x

#2 basis $u_1, u_2, \dots, u_n$ let $U = [u_1 \ u_2 \dots u_n]$

by uniq. coord: $x = d_1 u_1 + d_2 u_2 + \dots + d_n u_n$

$$x = U[d]_u \qquad [d]_u = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$ coord in U basis

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equal things are equal $(x = x)$

$$\boxed{V[c]_v = U[d]_u}$$

ex $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$

$[c]_v = \begin{bmatrix} .75 \\ .25 \end{bmatrix}_v$ $\qquad [d]_u = ?$

$$V[c]_v = U[d]_u$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .75 \\ .25 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}[d]_u$$

$$U^{-1} V [c]_V = [d]_U$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .75 \\ .25 \end{bmatrix} = [d]_U$$

Basis V to Basis U coord.

$$V [c]_V = U [d]_U$$

$$\boxed{U^{-1} V [c]_V = [d]_U}$$

Basis U to Basis V coord

$$\boxed{V^{-1} U [d]_U = [c]_V}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} .75 \\ .25 \end{bmatrix} = [d]_U$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 3/4 \\ 3 & 1 & 1/4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & -1/2 \end{array}\right]$$

.75
.25

$$.25 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-.5) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(See video)

.25

.75