Math 511

Q's 3.4 #7 Subspace S of $\mathbb{R}^4$ is $\begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix}$

$$= a\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\2\\0\\1\end{bmatrix}$$ ← linear combo of these (3) vectors

$$S = \text{Span}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right)$$

Basis? Dimension?

ex $\text{Span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}3\\3\\0\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$

$2v_1$ $3v_1$

Basis $= \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

dim $= 1$

back to this prob

$$\text{Span}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right)$$

Ind? Solve: $a\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Solve:
$a+b=0$
$a-b+2c=0$
$b=0$
$c=0$
→ $a=0$, $a=0$

only $a=0$, $b=0$, $c=0$ trivial soln.

So Ind.

So Basis $= \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

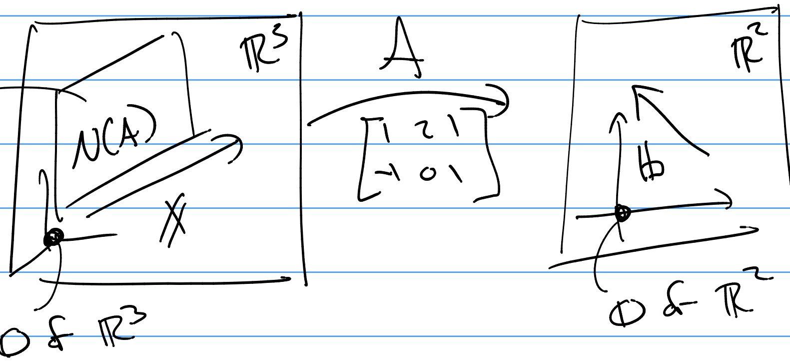
dim $= 3$

$A_{m\times n}$ $\qquad A\mathbf{x} = \mathbf{b}$

ex

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

$N(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$



$$N(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right]$$

x_3 is free

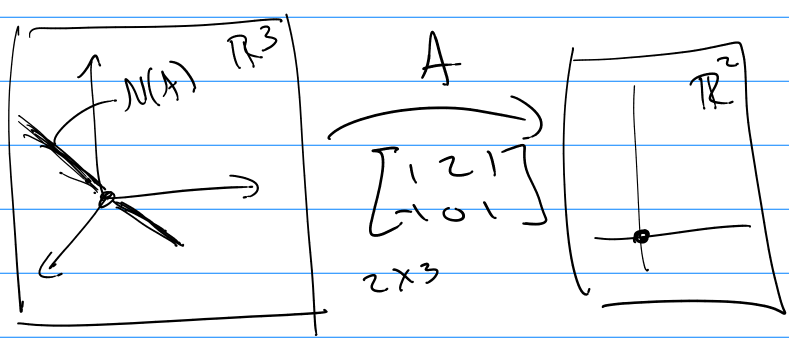
$x_3 = \alpha$
$x_2 = -\alpha$
$x_1 = \alpha$

$N(A)$ has $\to \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$N(A) = \text{Span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}\right)$ $\qquad \dim(N(A)) = \underline{1}$

$\quad\parallel$

$\underline{\# \text{ of free var's}}$



Def: Dimension of $N(A)$ is called A's nullity.

3.6 Study $\boxed{\text{rows of } A}$ $\boxed{\text{columns of } A}$

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

n-tuple   m-tuple

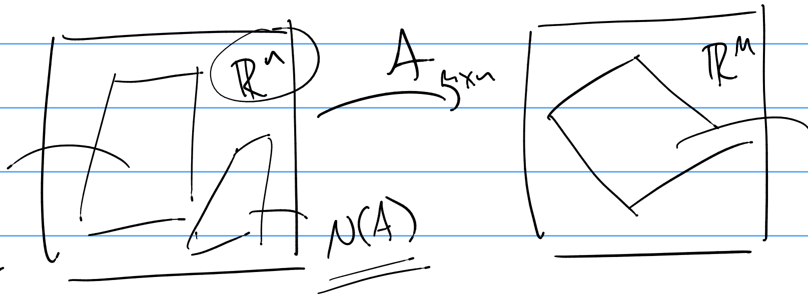
Consider: ① $\text{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m)$ is a subspace of $\mathbb{R}^n$ (b/c n-tuples)

↪ row space of A

② $\text{Span}(a_1, a_2, \ldots, a_n)$ is a subspace of $\mathbb{R}^m$ (b/c m-tuples)

↪ column space of A

Idea:



$\boxed{\text{Study Row Space of } A}$

↳ linear combination of A's rows.

Ex:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[r_2 + r_1 = nr_2]{\text{row ops}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

↳ u

thm two row-equiv. matrices have same row space.

ex $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ row space $A = \text{Span}([1,2,1], [-1,0,1])$

but

$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ row space $U = \text{Span}([1\ 0\ {-1}], [0\ 1\ 1])$

ex $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 4 & 3 \\ 0 & 4 & 4 \end{bmatrix}$ row space$(A) = ?$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

row space$(A) = \text{Span}([1\ 0\ {-1}], [0\ 1\ 1])$

dimension of row space$(A) = 2$

Def: dim of row space(A) = rank(A) = # of lead var's

remember nullity of A = dim of $N(A)$ = # of free var's

thm nullity(A) + rank(A) = n # of variables

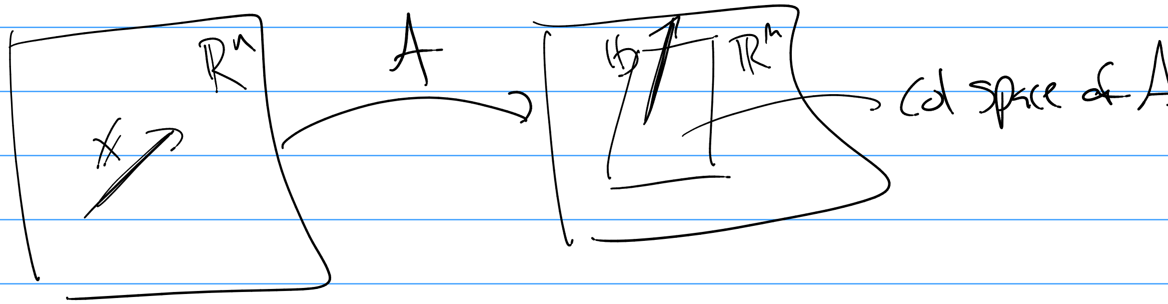
Study A's col's (col. space of A)

Note: $A\mathbf{x} \overset{?}{=} \mathbf{b}$ can be thought of as

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$

a linear combo of A's columns

Th^m $A\mathbf{x} = \mathbf{b}$ has a soln iff $\mathbf{b}$ is in A's col. space.



Th^m $A\mathbf{x} = \mathbf{b}$ is consistent for <u>all</u> $\mathbf{b} \in \mathbb{R}^m$ iff col. vectors of A span $\mathbb{R}^m$.

Th^m $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ and has at most one soln for each $\mathbf{b}$ <u>iff</u> col. vectors are <u>lin. ind.</u>

Cor $A_{n \times n}$ is non-singular <u>iff</u> col. vectors are a basis of $\mathbb{R}^n$.

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linear dep and A, U (reduced row esch. of A)

ex

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

U can tell us about $N(A)$, row space (all have been done) but U can tell us about linear relationships of col's.

$$U = \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \end{array} \qquad u_3 = -u_1 + u_2$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

thm linear dep. eqn's of U are same in A.

ex

$$u_3 = -u_1 + u_2$$
$$\underline{a_3} = \underline{-a_1 + a_2}$$

check: $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ $\qquad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{a_3} = \underbrace{\begin{bmatrix} -1 \\ +1 \end{bmatrix}}_{-a_1} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

So $\text{col.space}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\right)$ b/c lin. dep.

$$= \text{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

dim of col. space of A = 2 = rank(A)

Task: $\underset{m \times n}{A} \xrightarrow[\text{ops}]{\text{row}} U$ (red. row esch)