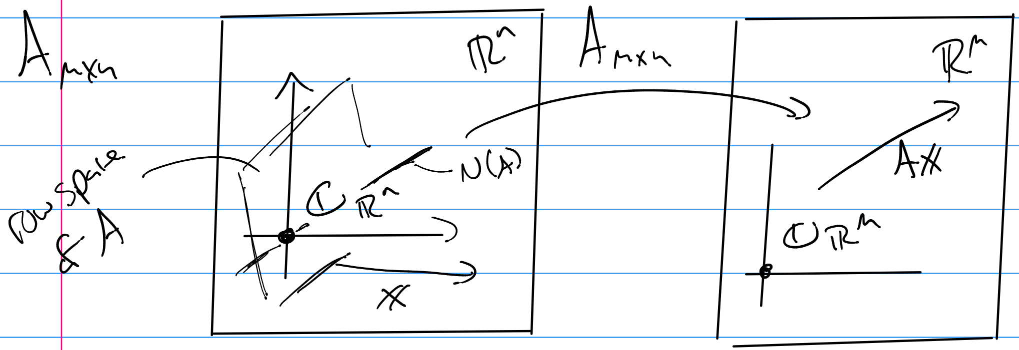
Math 511



take A

$\Big\}$ row ops

$\boxed{U}$ $\Big\}$ red. row ech.

① $N(A) = ?$

② row space of A

a) find its basis by looking at U taking non-zero rows as basis (write as cols)

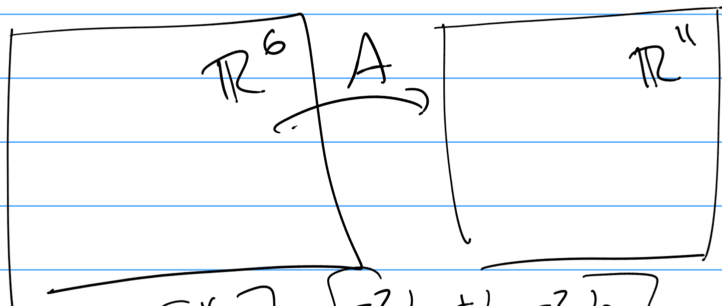
b) dim row space $=$ rank(A)

③ col space of A

a) look at U and find the lin. dep. eq'n's of it. and A has same lin. dep.

b) use above to find basis of col. space.

ex $\quad A_{4\times 6} \xrightarrow{\text{row ops}}$ $\quad U = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$



① $N(A)$ $\quad Ax = \mathbb{O}$

$Ux = \mathbb{O}$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha_3 \quad x_4 = \alpha_2 \quad x_6 = \alpha_1$

$x_5 = -\alpha_1$

$x_3 = -3\alpha_2 - \alpha_1$

$x_1 = -2\alpha_3 + \alpha_2 - 2\alpha_1$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2\alpha_1 + \alpha_2 - 2\alpha_3 \\ \alpha_3 \\ -\alpha_1 - 3\alpha_2 \\ \alpha_2 \\ -\alpha_1 \\ \alpha_1 \end{bmatrix}$$

$$X = \alpha_1 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ basis = $\{ [\ ], [\ ], [\ ] \}$

nullity of $A = 3$

② rowspace .. b/c $u = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

basis of row space is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

rank$(A) = 3$

③ col space: $u = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

find lin. dep. of cols.

$u_6 = 2u_1 + u_3 + u_5$

$u_4 = -u_1 + 3u_3$

$u_2 = 2u_1$

so $a_6 = \underline{2a_1 + a_3 + a_5}$

$a_4 = -a_1 + 3a_3$

$a_2 = 2a_1$

basis: $\{a_1, a_3, a_5\}$

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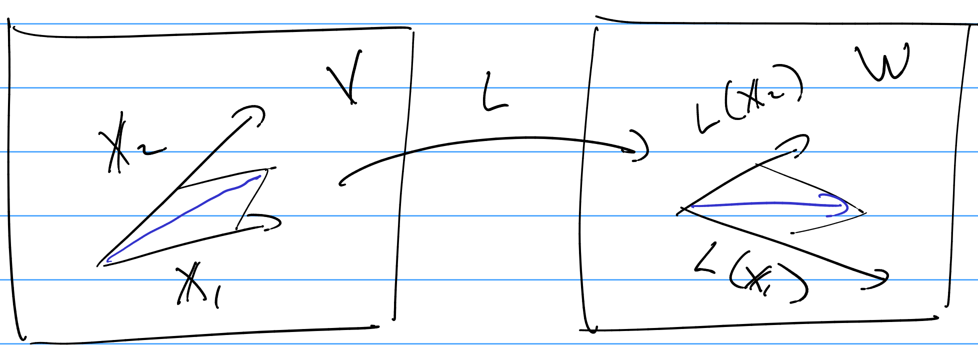
ch 4 Mappings from vector space V to vector space W that have "special" properties (call them Linear Transforms)



Def Mapping $L: V \to W$ is called a linear transform if

$$\underbrace{L(\alpha X_1 + \beta X_2)}_{\text{linear combo of pre-images}} = \underbrace{\alpha L(X_1) + \beta L(X_2)}_{\text{same linear combo of images.}}$$

or as a 2-step rule.

① $L(X_1 + X_2) = L(X_1) + L(X_2)$

② $L(\alpha X_1) = \alpha L(X_1)$

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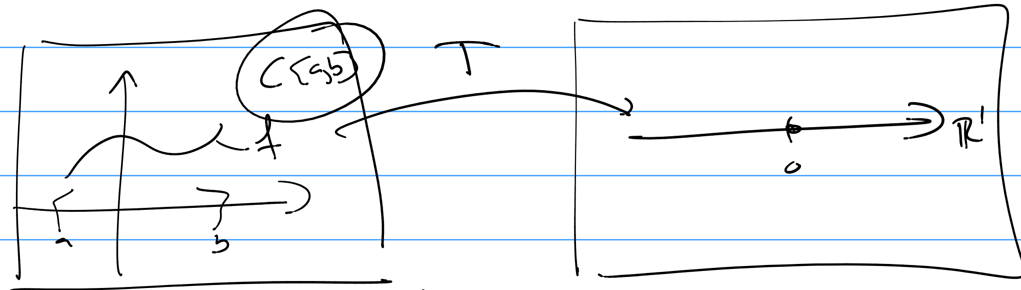
ex $T : C[a,b] \to \mathbb{R}^1$



$$T(f) = \int_a^b f(x)\,dx$$

$T(f)$ = area under f over $[a,b]$ (net signed area)

Is it a linear transform?

2 step:

① $L(X_1 + X_2) = L(X_1) + L(X_2)$?

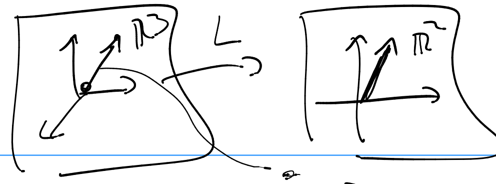
$$L(f_1 + f_2) = \int_a^b (f_1 + f_2)\,dx$$

$$\int_a^b f_1\,dx + \int_a^b f_2\,dx = L(f_1) + L(f_2) \checkmark$$

② $L(\alpha X_1) = \alpha L(X_1)$?

$$L(\alpha f_1) = \int_a^b \alpha f_1\,dx = \alpha \int_a^b f_1\,dx = \alpha L(f_1) \checkmark$$

ex $L: \mathbb{R}^3 \to \mathbb{R}^2$



$$\boxed{L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_3\\x_1+x_2\end{bmatrix}}$$

ex $L\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \cancel{\begin{bmatrix}1\\1+2\end{bmatrix}}$

$= \begin{bmatrix}1\\3\end{bmatrix}$

is it a linear transform?

check $L(\alpha_1 X_1 + \alpha_2 X_2) \overset{?}{=} \alpha_1 L(X_1) + \alpha_2 L(X_2)$

$$L\left(\alpha_1\begin{bmatrix}a\\b\\c\end{bmatrix} + \alpha_2\begin{bmatrix}d\\e\\f\end{bmatrix}\right) \overset{?}{=} \alpha_1 L\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) + \alpha_2 L\left(\begin{bmatrix}d\\e\\f\end{bmatrix}\right)$$

$$L\left(\begin{bmatrix}a\alpha_1 + d\alpha_2\\b\alpha_1 + e\alpha_2\\c\alpha_1 + f\alpha_2\end{bmatrix}\right) \overset{?}{=} \alpha_1\begin{bmatrix}c\\a+b\end{bmatrix} + \alpha_2\begin{bmatrix}f\\d+e\end{bmatrix}$$

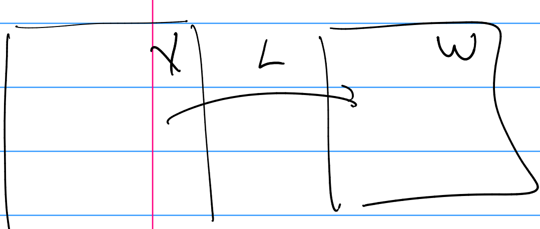
$$\begin{bmatrix}c\alpha_1 + f\alpha_2\\(a\alpha_1 + d\alpha_2) + (b\alpha_1 + e\alpha_2)\end{bmatrix} \overset{?}{=} \alpha_1\begin{bmatrix}c\\a+b\end{bmatrix} + \alpha_2\begin{bmatrix}f\\d+e\end{bmatrix}$$

$$\alpha_1\begin{bmatrix}c\\a+b\end{bmatrix} + \alpha_2\begin{bmatrix}f\\d+e\end{bmatrix} \overset{?}{=} \quad // \quad //$$

$\boxed{\text{Yes}}$

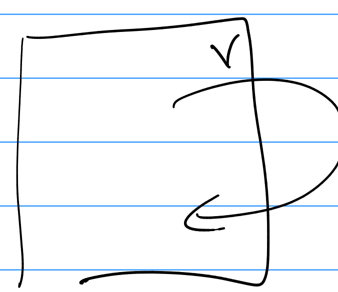
Properties of linear transforms $L: V \to W$

① if $L: V \to V$ call L a linear operator on V



Linear transform

vs



$L: V \to V$

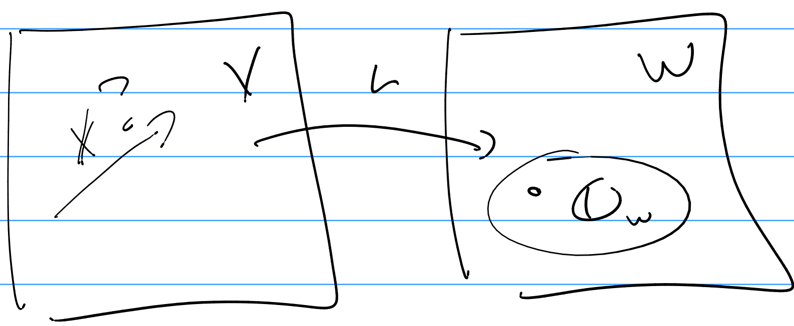
Linear operator

② $L(0_V) = 0_W$

③ $L(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n)$

$$= \alpha_1 L(x_1) + \alpha_2 L(x_2) + \dots + \alpha_n L(x_n)$$

④ $L(-x) = -L(x)$



⑤ Kernel of L $\quad \ker(L) = \{ v \in V \mid L(v) = 0_W \}$

⑥ S is a subspace of V. The image of S, $L(S)$

$\Rightarrow L(S) = \{ w \in W \mid \text{there exists } v \in S,\ L(v) = w \}$

5) $L(V)$ is range of L.

Th^m $\ker(L)$, $L(S)$ are subspaces

$\ker(L)$ is a subspace of V.

$L(S)$ is a subspace of W.

(ex) $D: P_2 \to P_2$ $\qquad$ P_2 is two term polynomials ($a + bx$)

D is the derivative.

(ex) $D(2 + \sqrt{3}x) = \sqrt{3} + 0 \cdot x$

$D(4) = 0 + 0 \cdot x$

1st: is D a linear operator? $L(\alpha_1 \underset{\text{in } P_2}{x_1} + \alpha_2 \underset{\text{in } P_2}{x_2}) \overset{?}{=} \alpha_1 L(x_1) + \alpha_2 L(x_2)$

$$D(\alpha_1(a+bx) + \alpha_2(c+dx)) \overset{?}{=} \alpha_1 D(a+bx) + \alpha_2 D(c+dx)$$

$$(\alpha_1 b + \alpha_2 d) + 0\cdot x \overset{?}{=} \alpha_1(b) + \alpha_2(d) + 0\cdot x$$

yes!

2nd: $\ker(D) \overset{?}{=}$

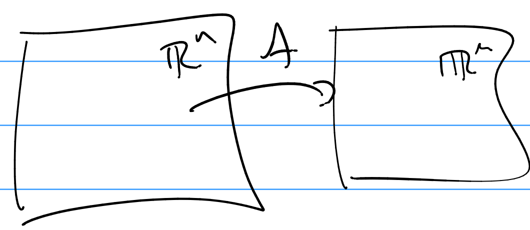
$$\ker(D) = \{ v \in V \mid Dv = \underset{0+0x}{0_w} \}$$

$$D(a+bx) \overset{?}{=} 0 + 0x$$

$$b + 0\cdot x \overset{?}{=} 0 + 0x \qquad \text{So } b=0$$

$$\ker(D) = \{ a + 0\cdot x \} = P_1$$

Consider: $A_{n\times n}$



Q? is Ax a linear transform? $L_A : \mathbb{R}^n \to \mathbb{R}^n$

check: $L_A(\alpha_1 x_1 + \alpha_2 x_2) \overset{?}{=} \alpha_1 L_A(x_1) + \alpha_2 L_A(x_2)$

so $A(\alpha_1 x_1 + \alpha_2 x_2) \overset{?}{=} \alpha_1 A x_1 + \alpha_2 A x_2$

$$A(\alpha_1 x_1) + A(\alpha_2 x_2) \overset{?}{=} \;\; ''$$

$$\alpha_1 A x_1 + \alpha_2 A x_2 = \text{yes!}$$

$\boxed{Th^m}$ if $L: \mathbb{R}^n \to \mathbb{R}^m$ is any liner transform

there exists a matrix A such that

$$L(\mathbb{X}) = A\mathbb{X}$$

$\underline{\underline{\text{sol:}}}$ $A = [a_{1}\ a_{2} \cdots a_{n}]$

$$A = [L(e_1)\ \ L(e_2)\ \cdots\ L(e_n)]$$

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ex $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_1 + x_2 \end{bmatrix}$

$$A = \left[L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$