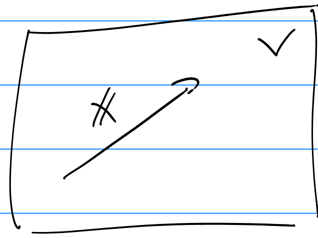
Math 511

Know: **Change of Basis:**

$\dim(V) = n$



(1) Standard Basis $\{e_1, e_2, \dots, e_n\}$

(2) Basis $B = \{b_1, b_2, \dots, b_n\}$

(3) Basis $D = \{d_1, d_2, \dots, d_n\}$

x in standard $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

To do: **Change of coord:** Matrix $B = [b_1 \; b_2 \dots b_n]$ $\quad D = [d_1 \; d_2 \dots d_n]$

a) $\boxed{[x] = B\,[x]_B}$

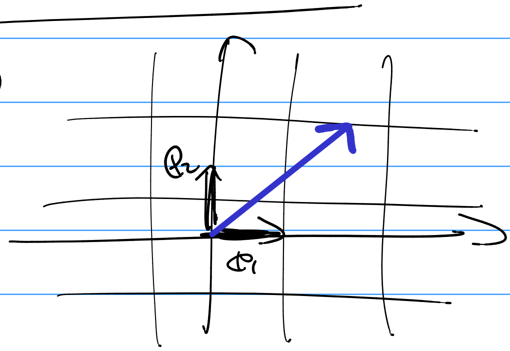
$[x]$: in standard coord. $\qquad [x]_B$: x's coord. in basis b_i

$x = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$

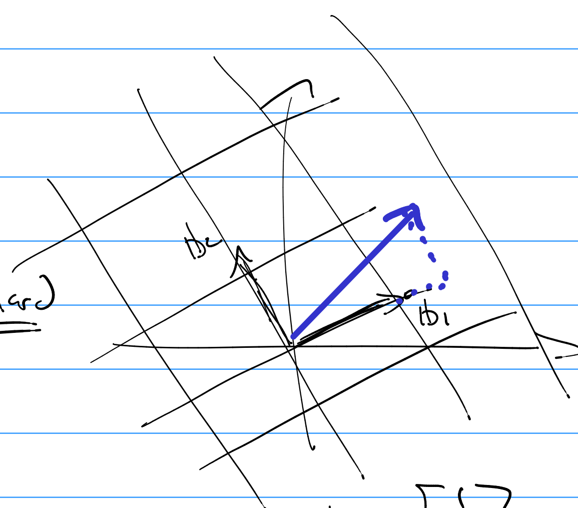
$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

b) $\boxed{[x]_B = B^{-1}[x]}$ $\quad$ ($[x]$: standard)

ex



standard



$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So $\quad B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ and $\quad [\;\;] = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} [\;\;]_B$

coor in standard $\qquad$ coord in B

and $\underset{\text{coord in } B}{[\ \ ]} = \begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}^{-1} \underset{\text{coord. in standard}}{[\ \ ]}$

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and if you have 2 Basis B, D

use $\boxed{D[x]_D = B[x]_B}$

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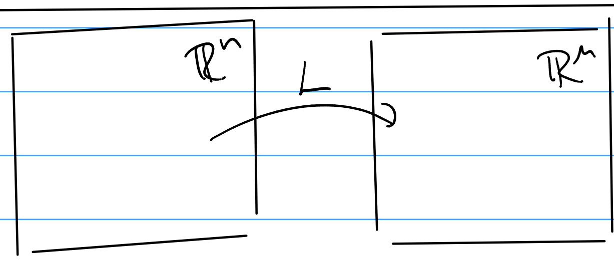
any $L: \mathbb{R}^n \to \mathbb{R}^m$ there is a matrix A such that

$$L(x) = Ax = y$$

and $A = [L(e_1)\ \ L(e_2)\ \dots\ L(e_n)]$



$L(x) = Ax$

$A = [L(e_1)\ \dots\ L(e_n)]$

ex $L\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right) = \begin{bmatrix}x_1\\ x_2\\ x_1+x_2\end{bmatrix}$ check $L(\alpha_1 x_1 + \alpha_2 x_2) =$

$L: \mathbb{R}^2 \to \mathbb{R}^3$ for linear trans $\alpha_1 L(x_1) + \alpha_2 L(x_2)$

$$A = \left[L\begin{pmatrix}1\\0\end{pmatrix}\ \ L\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$$

So $L(x) = \begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix} x$

ex $L\begin{pmatrix}5\\4\end{pmatrix} = \begin{bmatrix}5\\ 4\\ 5+4\end{bmatrix} = \begin{bmatrix}5\\4\\9\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix} = \begin{bmatrix}5\\4\\9\end{bmatrix}$

but what if ..

$$\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$$

given basis $B = [b_1 .. b_n]$ given basis $D = [d_1 .. d_m]$

want: ? Matrix for L that..

? $[y]_D = T\,[x]_B$ that does $L([x]_B) = [y]_D$

Put two ideas at start together.

① Standard Matrix representation of L

$$A = [L(e_1)\ \ L(e_2) ..\ \ L(e_n)]$$

$$\mathbb{R}^n \text{ Standard Basis} \xrightarrow{A} \mathbb{R}^m \text{ Standard Basis}$$

② $B = [b_1 .. b_n]$, x $D = [d_1 .. d_m]$, y

$$[y]_D = \underbrace{D^{-1} \underbrace{A \underbrace{B\,[x]_B}_{\text{gives } x \text{ in standard}}}_{\text{gives } [y] \text{ in standard}}}_{\text{gives } [y]_D}$$

③ So if you want one matrix for $L([X]_B) = [Y]_D$

L in standard

$$[Y]_D = \boxed{D^{-1} A B}\,[X]_B$$

let $T = D^{-1} A B$

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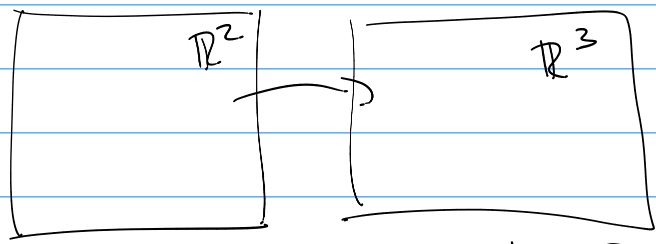
ex



$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

Bases: Standard $\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$ Standard $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

A) in standard L as $A = \left[L\begin{pmatrix}1\\0\end{pmatrix} \; L\begin{pmatrix}0\\1\end{pmatrix} \right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

B) in B's D's $[Y]_D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \underbrace{\underset{3\times 2}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}} [X]_B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -2 \\ 1 & -1 \\ 3 & 3 \end{bmatrix}$$

$[D \mid I]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$\downarrow$

$[I \mid D^{-1}]$

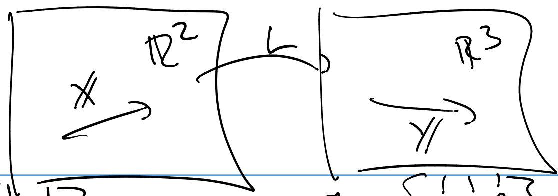
tech #1 find $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

tech #2 $\left[\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & -2 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{\text{row ops}} \left[I \;\middle|\; D^{-1} \begin{bmatrix} -1 & -2 \\ 1 & -1 \\ 3 & 3 \end{bmatrix} \right]$

let's do:

$$\left[\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & -2 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 1 & -4 & -5 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -4 & -5 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & -2 & -4 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right]$$

So  $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \\ x_1+2x_2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ $\qquad$ $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

in stand. to stand L as $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

in B to D coord L as $T = \begin{bmatrix} 3 & 3 \\ -2 & -4 \\ -2 & -1 \end{bmatrix}$

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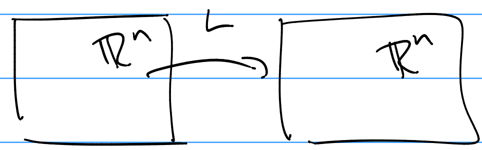
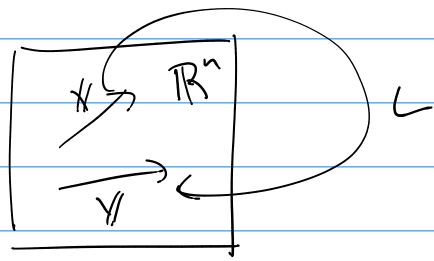
if:  on L on $\mathbb{R}^n$



$[y] = L([x])$

$A = [L(e_1) \dots L(e_n)]$

now: Standard $e_1, e_2, \dots e_n$

Basis $B = [b_1 \; b_2 \dots b_n]$

$$[y]_B = B^{-1} A B [x]_B$$

So L has two matrices to represent it

in standard $L([x]) = [y]$ is $[y] = A[x]$

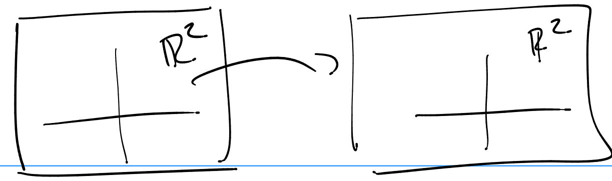
in basis B $L([x]_B) = [y]_B$ is $[y]_B = \underbrace{\boxed{B^{-1} A B}}_{= T} [x]_B$

Def: if $T = B^{-1} A B$

$A_{n\times n}$, $B_{n\times n}$, B is non-singular (B^{-1} exists)

call T, A similar

ex



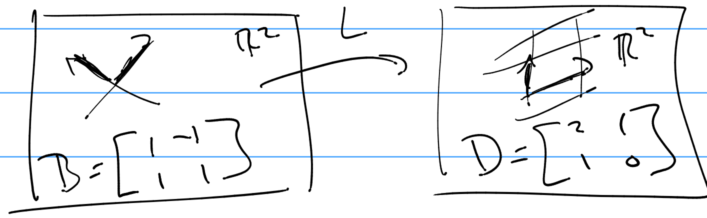
$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

Standard to standard: $L(\mathbb{X}) = A\,\mathbb{X}$

$$A = \left[\, L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \,\right] = \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

L in standard matrix

vs



$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[\mathbb{Y}]_D = D^{-1} A B [\mathbb{X}]_B$$

$$[\mathbb{Y}]_D = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{= S} [\mathbb{X}]_B$$

S is L from B coord to D coord.

a) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

check? $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $S = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$$