Math 511

$\boxed{Q5}$ S is a subspace of $C[a,b]$, $S = \text{Span}(\underbrace{e^x, xe^x, x^2e^x}_{\text{basis of } S})$

D is the differential operator

$$\text{Idea: } \underbrace{[y]}_{\text{coord of } y \text{ in given basis}} = L(\underbrace{[x]}_{\text{coord of } x \text{ in the given basis}})$$

ex coord $\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_3}$ $\quad [x] = c_1 e_1 + c_2 e_2 + c_3 e_3$

So for $S = \text{Span}(e^x, xe^x, x^2e^x)$

$$[x] = c_1 e^x + c_2 xe^x + c_3 x^2 e^x = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \begin{array}{l}\text{how many } e^x\\ xe^x\\ x^2e^x\end{array}$$

ex $\underbrace{\begin{bmatrix}-1\\0\\3\end{bmatrix}}_{A} = \boxed{(-1)e^x + (0)xe^x + (3)x^2e^x} = \underline{(3x^2-1)e^x}$

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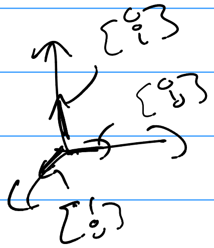
① $D(f)$ = takes its derivative. $= f'$

② A of L (a linear operator)

dim = 3



$$A = \left[\, \underline{L\begin{pmatrix}1\\0\\0\end{pmatrix}} \;\; L\begin{pmatrix}0\\1\\0\end{pmatrix} \;\; L\begin{pmatrix}0\\0\\1\end{pmatrix} \,\right]$$

Note $\boxed{\text{our problem}}$

$\begin{bmatrix}1\\0\\0\end{bmatrix} = e^x$

$\begin{bmatrix}0\\1\\0\end{bmatrix} = xe^x$

$\begin{bmatrix}0\\0\\1\end{bmatrix} = x^2e^x$

$\begin{bmatrix} \\ \end{bmatrix} = ?e^x + ?xe^x + ?x^2e^x$

So $\begin{bmatrix}1\\0\\0\end{bmatrix} = e^x$ transform = derivative. $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = D(e^x) = e^x = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\begin{bmatrix}0\\1\\0\end{bmatrix} = xe^x$ $L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = D(xe^x) = e^x + xe^x = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$\begin{bmatrix}0\\0\\1\end{bmatrix} = xe^x$ $L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = D(x^2e^x) = 2xe^x + x^2e^x = \begin{bmatrix}0\\2\\1\end{bmatrix}$

So
$e^x \xrightarrow{D} e^x$ $\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{D} \begin{bmatrix}1\\0\\0\end{bmatrix}$

$xe^x \xrightarrow{D} e^x + xe^x$ $\begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}1\\1\\0\end{bmatrix}$

$x^2e^x \xrightarrow{D} 2xe^x + x^2e^x$ $\begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\2\\1\end{bmatrix}$

$$\boxed{A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}}$$

Q) Subspace S is a subspace of V is ..

3.2 #3

1. S has $\mathbb{0}$
2. closure for $\mathbb{x}+\mathbb{y}$
3. closure for $\alpha\mathbb{x}$

$V = \mathbb{R}^{2\times2}$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$

3a) S is all 2X2 diagonal matrices

(i) $\mathbb{0} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is a diagonal matrix so $\boxed{\text{OK}}$

(ii) $\begin{bmatrix}a&0\\0&b\end{bmatrix} + \begin{bmatrix}c&0\\0&d\end{bmatrix} = \begin{bmatrix}a+c&0\\0&b+d\end{bmatrix}$ is diagonal so $\boxed{\text{OK}}$

(iii) $\alpha\begin{bmatrix}a&0\\0&b\end{bmatrix} = \begin{bmatrix}\alpha a&0\\0&\alpha b\end{bmatrix}$ is diagonal so $\boxed{\text{OK}}$

Exam 2 12 probs 110pts = 100%

3.1 Vector Space (Axioms) 1 prob

(i) Given axioms

(Q) Is (or Is not) V with $x+y$, αx a vector space.

check all 10 axioms.

ex $C[a,b]$ with $(f+g)(x) = f(x)+g(x)$ is a vector space.

$(\alpha f)(x) = \alpha f(x)$

I closure

(C1) $f \in C[a,b] \Rightarrow (\alpha f)(x) = \overbrace{\alpha \cdot f(x)}^{\text{real}\cdot\text{real}=\text{real}}$

is a continuous function by closure property of reals.

(C2) $f, g \in C[a,b]$

$\rightarrow (f+g)(x) = f(x)+g(x)$ is a cont. function by closure of addition.

II (A1) $f, g \in C[a,b]$

$(f+g)(x) \stackrel{?}{=} (g+f)(x)$

$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$

etc

yes

3.2 Subspaces (1 prob)

(1) See above (earlier in lecture)

3.3 Linear Independence (2 probs)

Note: $v_1, v_2, \ldots, v_k$ in V are linear ind. when

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0 \quad \text{Solve for } \alpha_i$$

only $\alpha_i = 0$ (all zeros soln) $\to$ linear ind.

vs

some $\alpha_i \neq 0$ soln (non-trivial soln) $\to$ lin. dependent.

① $\mathbb{R}^n$ space

② P_n space

$p_1(x) = 2x + 3$ in P_3

$p_2(x) = x^2 - x$

$p_3(x) = 2 - x^2$

$p_4(x) = 1 + x + x^2$

$$\begin{array}{c} p_1 \\ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{const} \\ x \\ x^2 \end{array} \begin{array}{c} p_2 \\ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} p_3 \\ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{array} \begin{array}{c} p_4 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

3.4 Basis Dimension (2 probs)

① given v_i in $\mathbb{R}^n$ $\to$ create a basis from them.

ex $\mathbb{R}^3$ — $\dim(\mathbb{R}^3) = 3$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{v_1 + v_2} \quad v_4 = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2 - v_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \boxed{1} & 0 & 1 & -1 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \underline{v_1 \text{ and } v_2 \text{ are ind.}}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$\left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & \pi \end{matrix} \right| = 1 \cdot 1 \cdot \pi = \pi \neq 0 \text{ so ind.}$$

② Basis for P_n

3.5 Change of Basis (1 prob)

① "Kernel for basis D, basis D $\boxed{D[x]_D} = \boxed{B[x]_B}$

so $[x]_D = D^{-1} B [x]_B$"

3.6 Row Space / Column Space (1 prob)

Given $A \to U$ find..

① $N(A)$ with basis (+) dimension

② rowspace with basis (+) dimension

③ col space with basis (+) dimension   (+) lin. dep. eqn's)

4.1 Linear Transforms (2 probs)

① is it a linear transform?

$$L(\alpha_1 x_1 + \alpha_2 x_2) \overset{?}{=} \alpha_1 L(x_1) + \alpha_2 L(x_2)$$

② $\ker(L)$

range (L)

4.2/4.3 (2 probs) both $\mathbb{R}^n$

① L as a matrix in standard $[y] = A[x]$

② L as a matrix for basis B.

$$[y]_B = B^{-1} A B [x]_B$$