Math 511

Exam fix for 50% missed points

ex -26 $\boxed{94}$

go into exam fix everything → 13 points back

Note: (1) $\mathbb{R}^2$

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$$

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha x_2 \end{bmatrix}$$

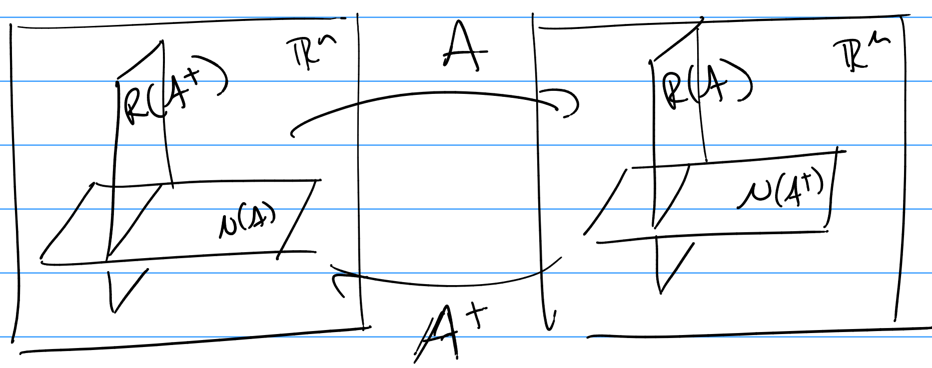
Note: $L = xp' + p''$ $\quad P_3$ $\quad$ any p in P_3

$$p = ax^2+bx+c = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

ex

$$L(p) = x(2ax+b) + (2a) = 2ax^2+bx+2a = \begin{bmatrix} 2a \\ b \\ 2a \end{bmatrix} = L\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Fundamental Subspaces $\quad A_{m\times n}$



$$R(A^T)^\perp = N(A) \qquad R(A)^\perp = N(A^T)$$

$$N(A)^\perp = R(A^T) \qquad N(A^T)^\perp = R(A)$$

$$\mathbb{R}^n = N(A) \oplus R(A^T) \qquad \mathbb{R}^m = N(A^T) \oplus R(A)$$

Thm $\dim(S) + \dim(S^\perp) = n$

$\quad\quad\quad\quad \overset{\shortparallel}{r} \quad\quad\quad \overset{\shortparallel}{n-r}$

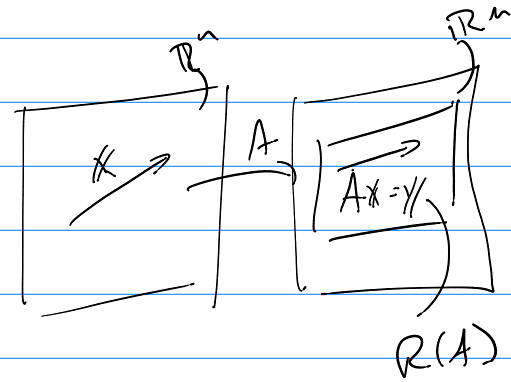
If Basis S is $\{x_1, x_2, \dots, x_r\}$

Basis $S^\perp$ is $\{x_{r+1}, \dots, x_n\}$

$\boxed{\text{Then}}$ $\{x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_n\}$ is a basis for $\mathbb{R}^n$

Applications of Above

① $Ax = y$ is a linear transform

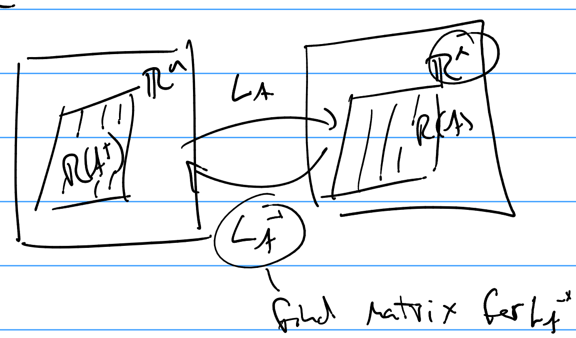


L_A : Domain $\to$ Codomain

if we strict to $R(A^T)$ in $\mathbb{R}^n$ for L_A

and $R(A)$ in $\mathbb{R}^m$

$L_A : \boxed{R(A^T)} \to R(A)$



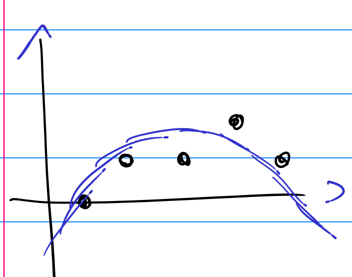
$\boxed{L_A}$ is one-to-one and onto

So L_A^{-1} exists

---

② $\boxed{5.3}$ Fit $(1,0), (2,1), (3,1), (4,2), (5,1)$

with $y = ax^2 + bx + c$



$(1,0)$ $\quad a(1)^2 + b(1) + c = 0$

$(2,1)$ $\quad a(2)^2 + b(2) + c = 1$

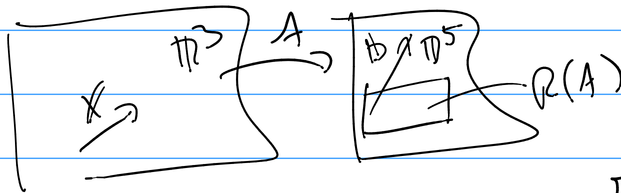
$(3,1)$ $\quad a(3)^2 + b(3) + c = 1$

$\vdots$

$a(1)^2 + b(1) + c = 0$
$a(2)^2 + b(2) + c = 1$
$a(3)^2 + b(3) + c = 1$
$a(4)^2 + b(4) + c = 2$
$a(5)^2 + b(5) + c = 1$

$$\longrightarrow \begin{bmatrix} 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \\ 3^2 & 3 & 1 \\ 4^2 & 4 & 1 \\ 5^2 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

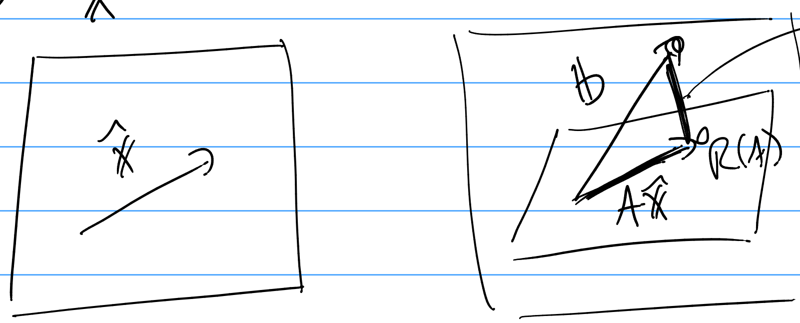
5×3



no solution b/c

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \notin R(A)$$

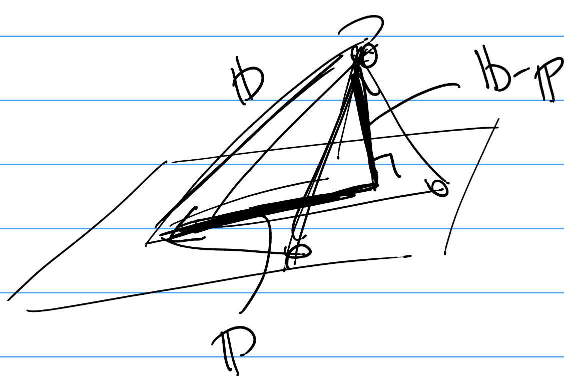
Goal: Find $\hat{x}$



$\|r(\hat{x})\| = \|b - A\hat{x}\|$ is a min.

Note: minimize $\|b - A\hat{x}\|$

Same place as minimize $\underline{\|b - A\hat{x}\|^2}$



minimize sums of squares of $(b - A\hat{x})$ components

$=$ least squares solution

$\boxed{\text{Th}^m}$ for any subspace S of $\mathbb{R}^m$ given any $\mathbb{b} \in \mathbb{R}^m$

there is a uniq. $\mathbb{p} \in S$ that is closest to $\mathbb{b}$

(means $\|\mathbb{b}-\mathbb{p}\| < \|\mathbb{b}-\mathbb{y}\|$ for any $\mathbb{y} \neq \mathbb{p}$)

and: $(\mathbb{b}-\mathbb{p}) \in S^{\perp}$

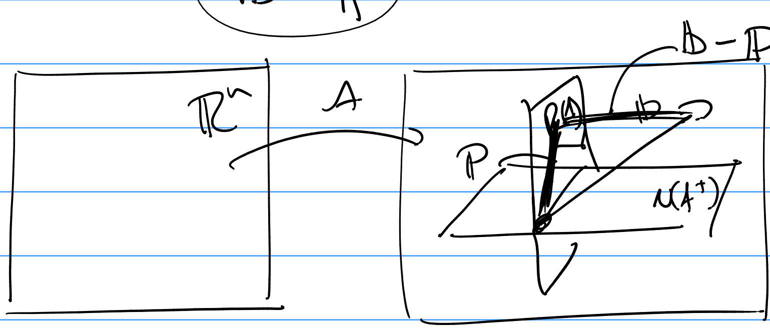
---

b/c it is true for any subspace -- let's use $R(A)$

$$\mathbb{p} \in R(A) \quad \text{closest to } \mathbb{b}$$

and

$$(\mathbb{b}-\mathbb{p}) \in R(A)^{\perp} = N(A^T)$$



so $A^T(\mathbb{b}-\mathbb{p}) = 0$

$$A^T\mathbb{b} = A^T\mathbb{p}$$

$\boxed{\text{Th}^m}$ Solve $\boxed{A\mathbb{x} = \mathbb{b}}$ but $\mathbb{b} \notin R(A)$ $\boxed{\text{No Soln}}$

Find least squares soln.

$$\boxed{A^TA\mathbb{x} = A^T\mathbb{b}}$$

Soln $\hat{\mathbb{x}} = (A^TA)^{-1}(A^T\mathbb{b})$

back to..

$$\begin{array}{cccccc} \begin{bmatrix}1^2&1&1\\2^2&2&1\\3^2&3&1\\4^2&4&1\\5^2&5&1\end{bmatrix} & \begin{bmatrix}a\\b\\c\end{bmatrix} & = & \begin{bmatrix}0\\1\\1\\2\\1\end{bmatrix} \end{array}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x\text{ data}^2 & x\text{ data}^1 & x\text{ data}^0 & y\text{ data} \end{array}$$

$$\underbrace{\begin{bmatrix}1&4&9&16&25\\1&2&3&4&5\\1&1&1&1&1\end{bmatrix}}_{3\times5}\underbrace{\begin{bmatrix}1&1&1\\4&2&1\\9&3&1\\16&4&1\\25&5&1\end{bmatrix}}_{5\times3}\begin{bmatrix}a\\b\\c\end{bmatrix}=\underbrace{\begin{bmatrix}1&4&9&16&25\\1&2&3&4&5\\1&1&1&1&1\end{bmatrix}}_{3\times5}\underbrace{\begin{bmatrix}0\\1\\1\\2\\1\end{bmatrix}}_{5\times1}$$

$$\underbrace{\begin{bmatrix}\ \end{bmatrix}}_{3\times3}\begin{bmatrix}a\\b\\c\end{bmatrix}=\underbrace{\begin{bmatrix}\ \end{bmatrix}}_{3\times1}$$

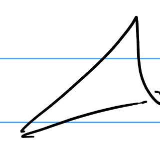
$x^T y = 0$

$\boxed{5.4}$ b/c orthogonal stuff in $\mathbb{R}^n$ seems useful.

But what about any vector space V?

Note: ① we can consider $x^T y$ to be a function. $f(x,y) = x^T y$

$$f(x,y) = f\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix},\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}\right) = x_1y_1 + x_2y_2 + \dots + x_ny_n$$



② Can we think of somethings like this for other V?

$\boxed{\text{Def}}$ call $\langle x, y\rangle$ for $x, y \in V$ an inner product

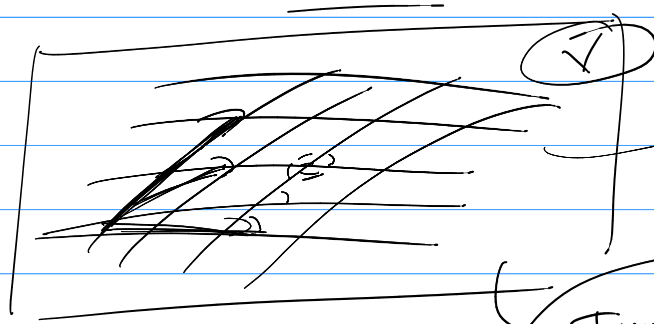
① $\langle x, x\rangle \geq 0$ and only $x = 0$ has $\langle x, x\rangle = 0$

② $\langle x, y\rangle = \langle y, x\rangle$

③ $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

Def: a Vector space V (Made up of objects and $v_1 + v_2$, αv_1)

with $\langle v_1 , v_2 \rangle$ inner product defined.



basis vectors

$V = \text{Span}(\text{basis vectors})$

Inner product space

$\langle v_1, v_2 \rangle$