Math 511

$\boxed{Q's}$ 5.2 #3

a) $S = \text{span}\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\}$ $\qquad A^T = B$

let $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$ $\quad B^T = \boxed{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = A}$

$N(A)^\perp = R(A^T) = \text{colspace of } \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = S$

$N(A)^\perp = S$ so $S^\perp = N(A)$

b) $\left(\text{span}\left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} \right) \right) = S$

$S^\perp = N\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \right)$

find who goes to 0

Solve $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right]$

$x_3 = \alpha$

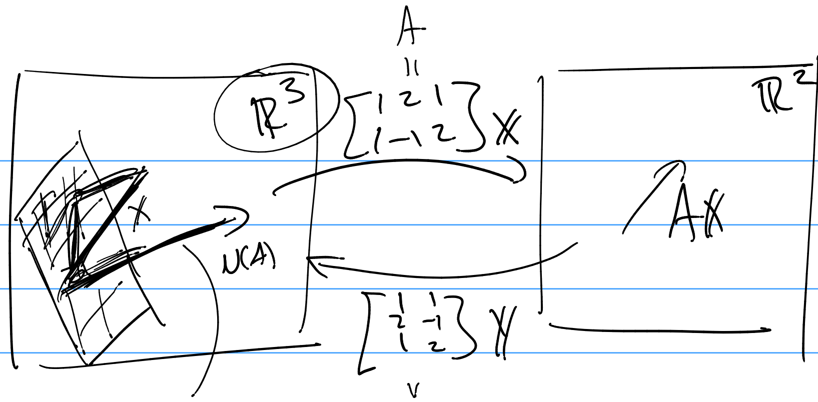
$x_2 = \frac{1}{3}\alpha$

$x_1 = -\frac{5}{3}\alpha$

$x = \begin{bmatrix} -\frac{5}{3}\alpha \\ \frac{1}{3}\alpha \\ \alpha \end{bmatrix}$

$x = \frac{\alpha}{3} \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

$N(A) = \text{span}\left(\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} \right)$



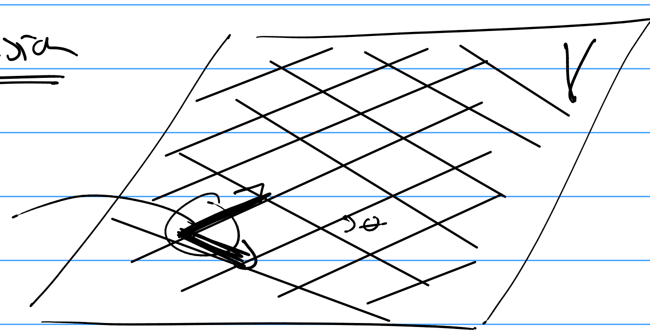
Span $\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

V is a vector space with αv, $v_1 + v_2$ defined

Spaces to know: $\mathbb{R}^n$, $\mathbb{R}^{m \times n}$, $C[a,b]$, P_n, P

Basis, Dimension

Basis



Inner Product: $\langle v_1, v_2 \rangle$ such that

① $\langle v_1, v_1 \rangle \geq 0$ and $\langle v_1, v_1 \rangle = 0$ iff $v_1 = 0$

② $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

③ $\langle \alpha v_1 + \beta v_2, v_3 \rangle = \alpha \langle v_1, v_3 \rangle + \beta \langle v_2, v_3 \rangle$

$x \quad y \quad z$

$\mathbb{R}^n$ (1) used $\langle X, Y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

Variation of above: given $w_1, w_2, \ldots, w_n$ all pos. scalars (weights)

$$\langle X, Y\rangle = w_1x_1y_1 + w_2x_2y_2 + \cdots + w_nx_ny_n$$

check: (1) $\langle X, X\rangle = w_1x_1^2 + w_2x_2^2 + \cdots + w_nx_n^2$ ✓

(2) ✓

(3) do check @ home (it is college algebra)

$\mathbb{R}^{m\times n}$

$$\langle A, B\rangle = \sum_j \sum_i a_{ij} b_{ij} = a_{11}b_{11} + a_{12}b_{12} + \cdots + a_{1n}b_{1n}$$
$$+ a_{21}b_{21} + a_{22}b_{22} + \cdots + a_{2n}b_{2n}$$
$$+ \vdots$$

check: properties this is an inner product

Note: $\|v\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = (\langle V, V\rangle)^{1/2}$

stay with this idea

$$\|A\| = \left\| \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix} \right\| = (\langle A, A\rangle)^{1/2}$$

$$= (1^2 + 2^2 + 3^2 + 0^2 + (-1)^2 + 4^2)^{1/2}$$

$$\|A\| = \sqrt{31}$$

Norm $\|A\| = (\langle A, A\rangle)^{1/2}$

$$\left\langle \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \right\rangle = -1 + 0 + 1 + 0 + -1 + 0$$
$$= -1$$

$C[a,b]$ ① $\langle f, g \rangle = \int_a^b f g \, dx$

② (weighted) $w(x) > 0$ $\langle f, g \rangle = \int_a^b f g w \, dx$

What can we do with $\langle v, u \rangle$ on V? — inner product spaces

#1 norm $\|v\| = \sqrt{\langle v, v \rangle}$

#2 $\langle v_1, v_2 \rangle = 0$ then $v_1 \perp v_2$

#3 Pythagorean Law if $x \perp y$

then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

#4 Projection u onto v

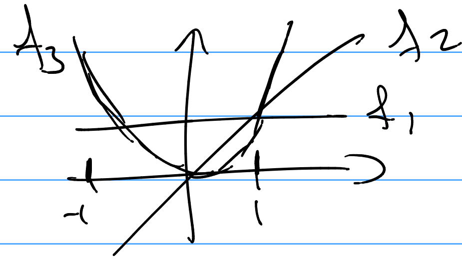
scalar projection $\alpha = \dfrac{\langle u, v \rangle}{\|v\|}$

Vector proj. of u onto v $\quad p = \alpha \dfrac{1}{\|v\|} v = \dfrac{\langle u, v \rangle}{\langle v, v \rangle} v$

and $u - p \perp p$

ex $\quad C[-1,1] \qquad \langle f,g\rangle = \int_{-1}^{1} fg\,dx$

consider: $\quad f_1(x) = 1$
$\qquad\qquad f_2(x) = x$
$\qquad\qquad f_3(x) = x^2$



$$\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = 0 \qquad f_1 \perp f_2$$

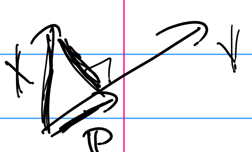
$$\langle 1, x^2\rangle = \int_{-1}^{1} 1\cdot x^2\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

$$\langle x, x^2\rangle = \int_{-1}^{1} x\cdot x^2\,dx = 0 \qquad f_2 \perp f_3$$

$$\|f_1\| = (\langle f_1, f_1\rangle)^{1/2} = \sqrt{\int_{-1}^{1} 1\cdot 1\,dx} = \sqrt{2}$$

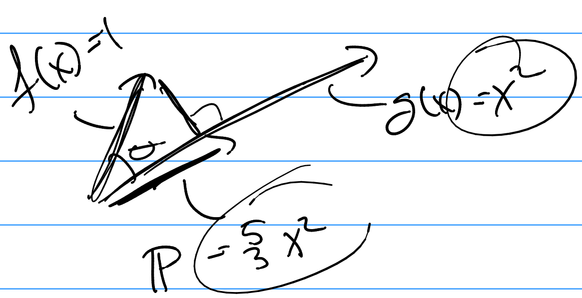
$$\|f_3\| = (\langle x^2, x^2\rangle)^{1/2} = \sqrt{\int_{-1}^{1} x^4\,dx} = \sqrt{2/5}$$

Project $\quad f(x) = 1$ onto $g(x) = x^2$



$$\alpha = \frac{\langle f, g\rangle}{\|g\|} = \frac{\langle 1, x^2\rangle}{\|x^2\|} = \frac{2/3}{\sqrt{2/5}} \quad \frac{\sqrt{2}}{3}\cdot\frac{\sqrt{5}}{\sqrt{2}} = \frac{1}{3}\sqrt{10}$$

$$P = \alpha\,\frac{1}{\|g\|}\,g = \frac{1}{3}\sqrt{10}\,\frac{1}{\sqrt{2/5}}\,x^2 = \frac{1}{3}\,\frac{\sqrt{10}}{1}\,\frac{\sqrt{5}}{\cancel{\sqrt{2}}}\,x^2$$
$$= \frac{5}{3}x^2$$



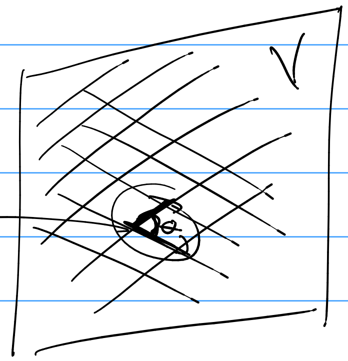
Properties

#5 Cauchy-Schwartz $\quad |\langle u, v\rangle| \le \|u\|\,\|v\|$

Says $\quad -1 \le \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|} \le 1$

So.. uniq. θ such that $\quad \cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

$\theta_{x,y}$



$$\theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\,\|v\|}\right)$$

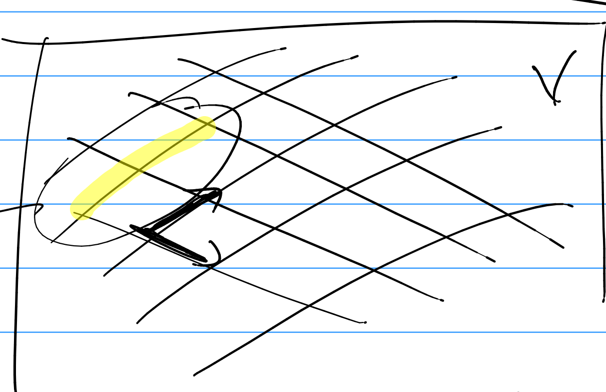
above: $\quad \langle 1, x^2\rangle = \int_{-1}^{1} 1\cdot x^2\,dx = \frac{2}{3}$

$$\|1\| = \sqrt{\int_{-1}^{1} 1^2\,dx} = \sqrt{2}$$

$$\|x^2\| = \sqrt{\int_{-1}^{1} x^4\,dx} = \sqrt{2/5}$$

$\angle$ θ between $f_1(x) = 1$ $\quad f_2(x) = x^2$ $\qquad \cos\theta = \dfrac{2/3}{\sqrt{2\cdot 2/5}}$

Study Norms a bit more



Metric for magnitude? Norm

we already have $\|v\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$ for $\mathbb{R}^n$

Def $\|v\|$ is a Norm and vector space V with the norm is called a Normed Linear Space if

① $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
② $\|\alpha v\| = |\alpha| \|v\|$
③ $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

Variations on $\mathbb{R}^n$

b/c $\|v\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$

vs $\|v\|_p = (|v_1|^p + |v_2|^p + \dots + |v_n|^p)^{1/p}$ p-norm

So old $\|v\| = \|v\|_2$

So $p \to \infty$ $\|v\|_\infty = \max |v_i|$

$\|v\|_1 = |v_1| + |v_2| + \dots + |v_n|$