Math 511

HW 9, 10 Due May 7

5.5

Note: $\{b_1, b_2, \ldots, b_k\}$ $\quad v = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}_B$$

coordinates of v in basis B

ex) $\{1, 1+x, \sin x\} = B$

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}_B = 3(1) + 2(1+x) + (-1)(\sin x) = 5 + 2x - \sin x$$

ex) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = B$

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_B = -1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}_E$$

5.5 Orthonormal Sets (inner product spaces)

vector space V with $\langle v_1, v_2 \rangle$ def.

Def $v_1, \ldots, v_k$ if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$

then the set $v_1, \ldots, v_k$ are an orthogonal set of vectors.

Def $v_1, \ldots, v_k$ are an orthogonal set

AND if $\|v_i\| = 1$

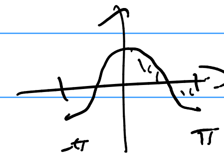
then the set $v_1, \ldots, v_k$ is an orthonormal set.

ex $V = C[-\pi, \pi]$ <u>def:</u> $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg \, dx$

consider $\{1, \sin x, \cos x\}$ $\quad V_1 = 1$

$V_2 = \sin x$

$V_3 = \cos x$



(1) <u>orthogonal?</u>

$$\langle V_1, V_2 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \, dx = 0$$

$$\langle V_1, V_3 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \, dx = 0$$

$$\langle V_2, V_3 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \cos x \, dx = 0$$

(2) <u>orthonormal?</u> already know orthogonal so just check $\|V_i\| = 1$

$$\|V\| = \sqrt{\langle V, V \rangle}$$

$$\|V_1\| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} 1 \, dx} = \sqrt{2} \leftarrow \text{not orthonormal}$$

$$\|V_2\| = \sqrt{\frac{1}{\pi}\boxed{\int_{-\pi}^{\pi} \sin^2 x \, dx}} = 1$$

$$\|V_3\| = \sqrt{\frac{1}{\pi}\boxed{\int_{-\pi}^{\pi} \cos^2 x \, dx}} = 1$$

<u>But</u> $\left\{ \frac{1}{\sqrt{2}}, \sin x, \cos x \right\}$ <u>is</u> an orthonormal set

so $4 + \sin x - \cos x = 4\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + 1(\sin x) + (-1)(\cos x)$

$$= \begin{bmatrix} 4\sqrt{2} \\ 1 \\ -1 \end{bmatrix}_B$$

Thm if $\{v_1, .., v_k\}$ are an orthogonal set then they are linearly independent.

So...

given $b_1, .., b_n$ an orthonormal set.

then $S = \text{Span}(b_1, .., b_n)$ has $b_1, .., b_n$ as it's orthonormal basis.

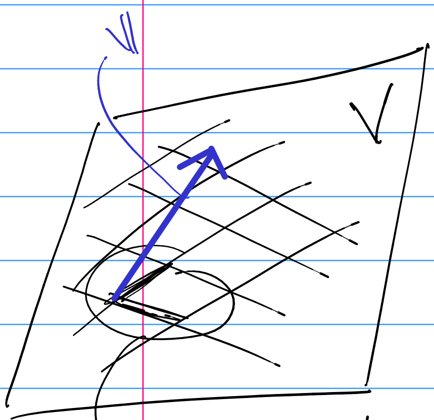
(ex) $\{\frac{1}{\sqrt{2}}, \sin x, \cos x\}$ $\quad S = \text{Span}(\frac{1}{\sqrt{2}}, \sin x, \cos x)$

$S = c_1 \frac{1}{\sqrt{2}} + c_2 \sin x + c_3 \cos x$

Thm given $u_1, .., u_n$ an orthonormal basis for Inner Product Space.

if $v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$

$v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_U \leftarrow$ coord. of v in the orthonormal basis U.



basis: orthonormal

then: $c_i = \langle v, u_i \rangle$

(we can find the coord. in orthonormal basis U simply using $\langle , \rangle$)

(ex) $\{\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x\}$ is orthonormal

$v = 2 + 4\cos x - 6\sin 2x$

$= c_1\left(\frac{1}{\sqrt{2}}\right) + c_2(\sin x) + c_3(\cos x) + c_4(\sin 2x) + c_5(\cos 2x)$

$$c_1 = \left\langle v, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (2+4\cos x - 6\sin 2x)\frac{1}{\sqrt{2}}\,dx$$

$$c_2 = \langle v, \sin x \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (2+4\cos x - 6\sin 2x)(\sin x)\,dx$$

⋮
etc

we can find the coord... Why?

Th^m u_i are an orthonormal basis

$$a = a_1 u_1 + a_2 u_2 + \dots + a_n u_n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \leftarrow \text{coord}$$

$$b = b_1 u_1 + b_2 u_2 + \dots + b_n u_n = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \leftarrow$$

Then $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

Ex $\left\{ \frac{1}{\sqrt{2}}, \sin x, \cos x \right\} = B$

$$a = 6 - 2\sin x + \cos x = \begin{bmatrix} 6\sqrt{2} \\ -2 \\ 1 \end{bmatrix}_B$$

$$b = 3\cos x - \sin x + 1 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 3 \end{bmatrix}_B$$

$$\langle a, b \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (6 - 2\sin x + \cos x)(3\cos x - \sin x + 1)\,dx$$

$$= (6\sqrt{2})(\sqrt{2}) + (-2)(-1) + (1)(3) = \boxed{17}$$

by th^m

Corollary (Parseval's Formula)

$$a = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

$$\|a\|^2 = \langle a, a \rangle = a_1^2 + a_2^2 + \cdots + a_n^2$$

ex $\{ \frac{1}{\sqrt{2}}, \sin x, \cos x \} = B$

$$a = (6 - 2\sin x + \cos x)^2 = \begin{bmatrix} 6\sqrt{2} \\ -2 \\ 1 \end{bmatrix}_B$$

$$b = 3\cos x - \sin x + 1 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 3 \end{bmatrix}_B$$

$$\|a\|^2 = (6\sqrt{2})^2 + (-2)^2 + (1)^2 = 77$$
$$\|b\|^2 = (\sqrt{2})^2 + (-1)^2 + (3)^2 = 12$$

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if we let $b_1, b_2, \ldots, b_n$, $b_i \in \mathbb{R}^n$ orthonormal set

$B = [b_1\ b_2 \ldots b_n]$ is $n \times n$ matrix

we call B an orthogonal matrix.

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ex $Q = [q_1, q_2 \ldots q_n]$ is an orthogonal matrix

$$Q^T Q = [q_1, q_2 \ldots q_n]^T [q_1, q_2 \ldots q_n]$$

$$= \begin{bmatrix} \overset{=1}{q_1^T q_1} & \overset{=0}{q_1^T q_2} & \cdots & \overset{=0}{q_1^T q_n} \\ q_2^T q_1 & \overset{=1}{q_2^T q_2} & \cdots & \\ \vdots & & \ddots & \\ & & & \overset{=1}{q_n^T q_n} \end{bmatrix}$$

b/c q_i are orthonormal

$q_i^T q_j = 0$ $i \neq j$

$$= I$$

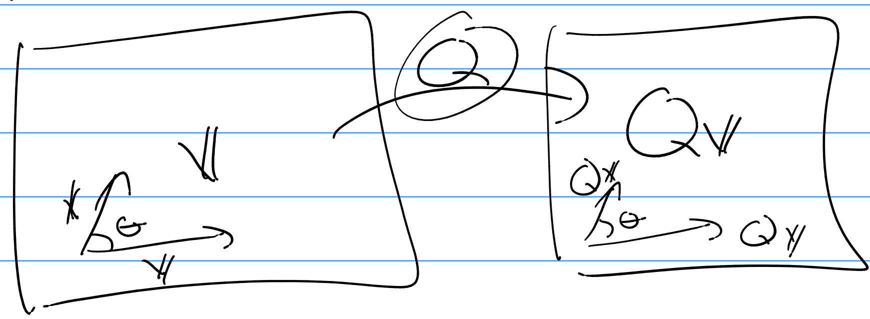
so $Q^TQ = I$ says $Q^T = Q^{-1}$

Properties of Q an orthogonal matrix

① $Q = [q_1, q_2, \dots, q_n]$ q_i are an orthonormal basis for $\mathbb{R}^n$

② $Q^TQ = I$

③ $Q^T = Q^{-1}$

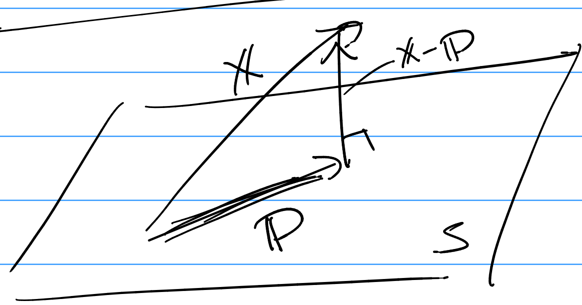


$$\langle Qx, Qy\rangle = (Qx)^T(Qy) = x^T \underbrace{Q^TQ}_{=I} y = x^Ty = \langle x, y\rangle$$

④ $\langle Qx, Qy\rangle = \langle x, y\rangle$

⑤ $\|Qx\|_2 = \sqrt{\langle Qx, Qx\rangle} = \sqrt{(Qx)^T(Qx)} = \sqrt{x^TQ^TQx} = \sqrt{x^Tx} = \|x\|_2$

Another Application of orthonormal basis



$x \notin S$

$p = c_1 u_1 + \dots + c_k u_k$

$c_i = \langle x, u_i\rangle$

$S = \mathrm{Span}(\underbrace{u_1, \dots, u_k}_{\text{orthonormal basis}})$

(ex) $S = \mathrm{Span}\left(\frac{1}{\sqrt{2}}, \sin x, \cos x\right)$ (vs) $x = 3x^2 + 2$

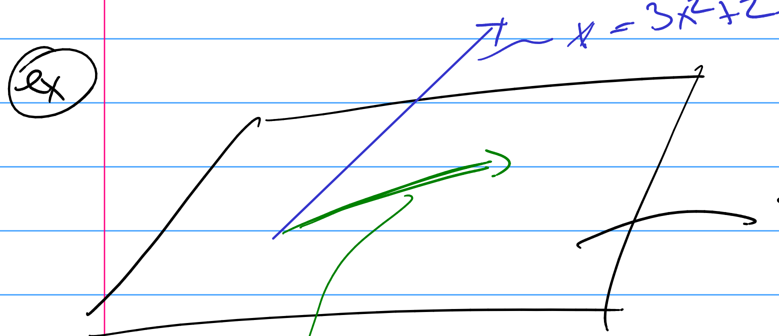
Th^n $\quad X - P \in S^{\perp}$

So $P = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$

$c_i = \langle X, u_i \rangle$

is the closest object in S to X.

ex



$\text{Span}\left(\frac{1}{\sqrt{2}}, \sin x, \cos x\right)$

$V = c_1 \frac{1}{\sqrt{2}} + c_2 \sin x + c_2 \cos x$

$P = \boxed{c_1}\left(\frac{1}{\sqrt{2}}\right) + \boxed{c_2}(\sin x) + \boxed{c_3}(\cos x)$

$c_1 = \langle 3x^2+2, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} 3x^2+2 \, dx = \square$

$c_2 = \langle 3x^2+2, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2+2) \sin x \, dx = \square$

$c_3 = \langle 3x^2+2, \cos x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2+2) \cos x \, dx = \square$

Fourier Transform

orthonormal set $\quad \frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin nx, \cos nx$

$V = c_1 \frac{1}{\sqrt{2}} + c_2 \sin x + c_3 \cos x + c_4 \sin 2x + c_5 \cos 2x + \dots + c_{2n} \sin nx + c_{2n+1} \cos nx$