Math 511

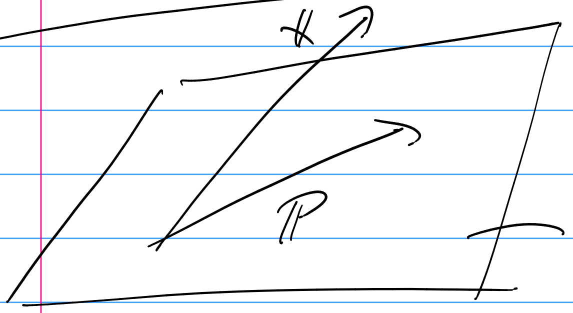
$\boxed{Q's}$ data fitting : (ex) $(0.2, 1), (4,2), (-1,3), (\frac{1}{2}, -1)$

with what? Lines $y = ax + b$

$\boxed{y = a + bx}$

$(0.2, 1) \rightarrow 1 = a + 0.2b$

$(4, 2) \rightarrow 2 = a + 4b$ $\rightarrow$ $\underbrace{\begin{bmatrix} 1 & 0.2 \\ 1 & 4 \\ \vdots & \vdots \end{bmatrix}}_{} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

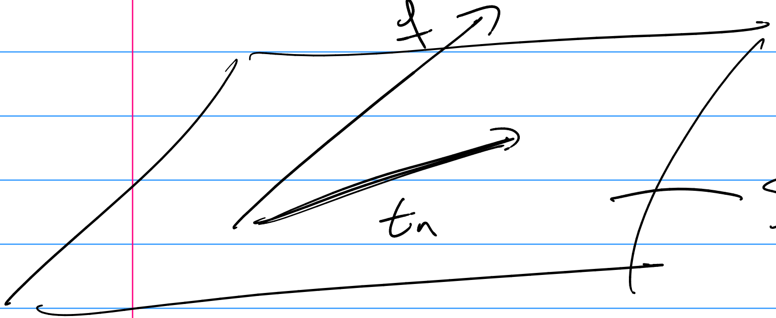


orthonormal basis $u_1, u_2 .., u_k$

$P = \underline{\underline{\langle x, u_1\rangle}} u_1 + \underline{\underline{\langle x, u_2\rangle}} u_2 + \cdots + \underline{\underline{\langle x, u_k\rangle}} u_k$

Fourier Transform

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg \, dx$$



Span $\frac{1}{\sqrt{2}}$, $\cos x$, $\sin x$

$\cos 2x$, $\sin 2x$

$\vdots$

$\cos nx$, $\sin nx$

goal $\underline{\underline{f \approx f_n}}$

$f_n = \underline{\underline{\langle f, \frac{1}{\sqrt{2}}\rangle}} \frac{1}{\sqrt{2}} + \Big[\underline{\underline{\langle f, \cos x\rangle}} \cos x + \underline{\underline{\langle f, \sin x\rangle}} \sin x + \ldots$

Note: $\langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cdot \frac{1}{\sqrt{2}}\, dx$

$\langle f, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} = \frac{1}{2} \frac{1}{\pi} \int_{-\pi}^{\pi} f \cdot 1\, dx = \frac{1}{2} \langle f, 1 \rangle$

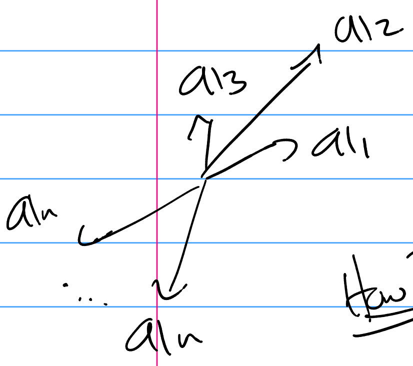
Let $f_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$

$a_0 = \langle f, 1 \rangle = \int_{-\pi}^{\pi} f\, dx$

$a_k = \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$

$b_k = \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$

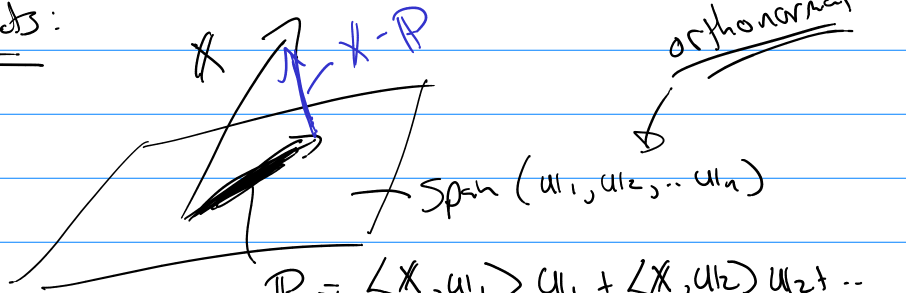
$\boxed{5.6}$ given $u_1, u_2, u_3, \dots, u_n$ basis. $\boxed{\text{But}}$ not orthonormal



$\boxed{Q}$ Can I create a new basis $q_1, q_2 \dots q_n$ that is orthonormal by using u_i ?

How? two facts:

#1

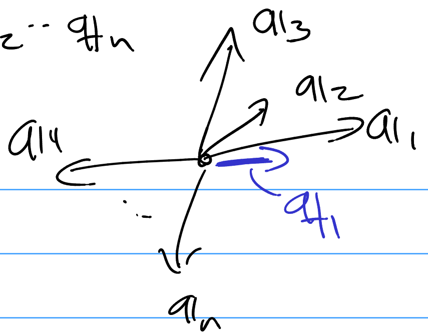


$P = \langle X, u_1 \rangle u_1 + \langle X, u_2 \rangle u_2 + \dots + \langle X, u_n \rangle u_n$

#2 $X - P \in S^{\perp}$

So $X - P$ is orthogonal to $u_1, u_2, \dots, u_n$

go from $a_1, a_2, \dots a_n$ to $q_1\ q_2 \cdots q_n$

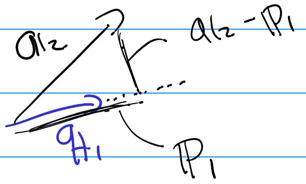


Gram - Schmidt Process

① **Base Step** $q_1 = \frac{1}{\|a_1\|} a_1$

Inductive Steps: (make q_2 then q_3 then q_4 -- q_n)

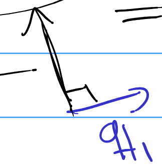
② ($a_2 \to q_2$)



a) project a_2 onto q_1

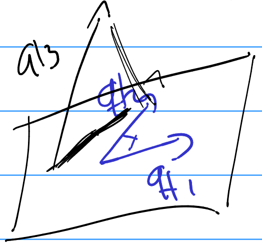
$$P_1 = \boxed{\langle a_2, q_1\rangle} \underline{q_1}$$

b) (⊥) $\underline{a_2 - P_1}$



c) (normalize) $q_2 = \frac{1}{\underline{\|a_2 - P_1\|}} (a_2 - P_1)$

③ ($a_3 \to q_3$)



a) project a_3 onto known q_1, q_2

$$P_2 = \underline{\langle a_3, q_1\rangle} q_1 + \underline{\langle a_3, q_2\rangle} q_2$$

b) (⊥) $a_3 - P_2$

c) (normalize) $q_3 = \frac{1}{\|a_3 - P_2\|} (a_3 - P_2)$

⋮

(k) ($a_k \to q_k$)

a) **project:** $P_{k-1} = \langle a_k, q_1\rangle q_1 + \langle a_k, q_2\rangle q_2 + \dots + \langle a_k, q_{k-1}\rangle q_{k-1}$

b) (⊥) $a_k - P_{k-1}$

c) (normalize) $q_k = \frac{1}{\|a_k - P_{k-1}\|} (a_k - P_{k-1}))$

Note: $A = [a_1\ a_2 \cdots a_n] \to \underset{\text{orthogonal matrix}}{Q} = [\overset{\text{all orthonormal}}{q_1\ q_2 \cdots q_n}]$

Note: $[a_1\ a_2 \cdots a_n] = [q_1\ q_2 \cdots q_n] \left[\begin{array}{cccc} \|a_1\| & \langle a_2, q_1\rangle & \langle a_3, q_1\rangle & \\ 0 & \|a_2 - p_1\| & \langle a_3, q_2\rangle & \cdots \\ 0 & 0 & \|a_3 - p_2\| & \\ \vdots & \vdots & 0 & \\ 0 & 0 & \vdots & \\ \text{assoc. } a_1 & \text{assoc. } a_2 & \text{assoc. } a_3 & \end{array}\right]$

So $A = \underset{\text{orthogonal}}{Q}\ \underset{\text{upper triangular}}{R}$

Before: we could find $A = LU$ (use type 3 row ops)

Now we can also find $A = QR$ (use gram schmidt

why QR? **consider** $Ax = b$

$+ A = QR$

$QRx = b$

$Q^TQRx = Q^Tb$

Q orthogonal says $Q^{-1} = Q^T$

$$\boxed{Rx = Q^Tb}$$

can be quickly solved by using back substitution ex $A = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 1 & 4 & 1 \\ 2 & 3 & 2 \\ -2 & 2 & 1 \end{array}$ $Q = \begin{array}{ccc} q_1 & q_2 & q_3 \\ \hline 1/3 & 2/3 & \boxed{} \\ 2/3 & 1/3 & \\ -2/3 & -2/3 & \end{array}$ $R = \begin{bmatrix} \boxed{3} & \boxed{2} & \boxed{} \\ 0 & \boxed{5} & \boxed{} \\ 0 & 0 & \boxed{} \end{bmatrix}$

Basis: ① $q_1 = \frac{1}{\|a_1\|} a_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

Inductive ② $(a_2 \to q_2)$ a) $P_1 = \langle a_2, q_1 \rangle q_1 = 2\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 \\ 4/3 \\ -4/3 \end{bmatrix}}$

b) $(\perp)$ $a_2 - P_1 = \begin{bmatrix} 10/3 \\ 5/3 \\ 10/3 \end{bmatrix}$

c) $q_2 = \frac{1}{5}\begin{bmatrix} 10/3 \\ 5/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

etc