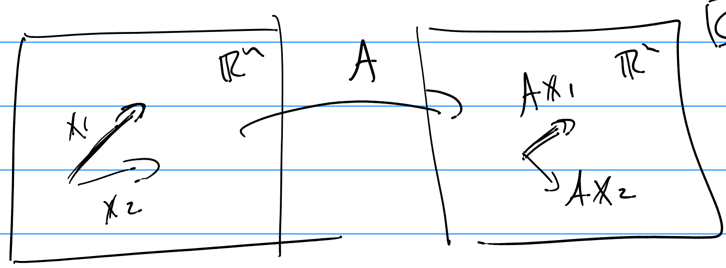
Math 511

Ch6 Eigen values / Eigen vectors



Q: does $Ax = \lambda x$
scalar "stretch"

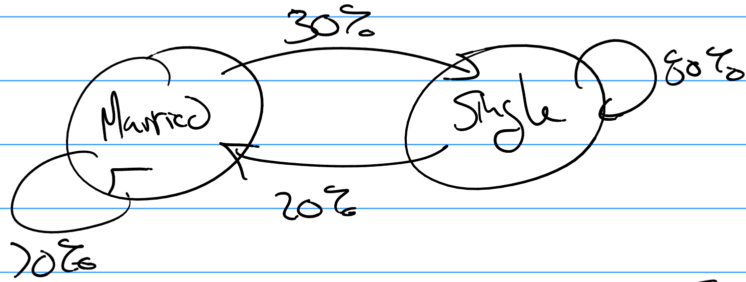
(Linear transform, A, doesn't change x's direction just a stretch / compression / flip)?

why? #1 Markov process

x_0, $x_1 = Ax_0$, $x_2 = Ax_1 = A^2 x_0$, $x_3 = Ax_2 = A^3 x_0$

... Markov chain $x_0, x_1, x_2, \ldots, x_k = A^k x_0$

ex



percentage of (what?)

70% M + 20% S = M
30% M + 80% S = S

$$\underbrace{\begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} M \\ S \end{bmatrix}}_{x_0} = \underbrace{\begin{bmatrix} M \\ S \end{bmatrix}}_{x_1}$$

Idea: $Ax = \lambda x$ for some λ, x pair

eigen value
eigen vector

Solve? set one side to 0

$Ax - \lambda x = 0$ but $Ix = x$

$Ax - \lambda(Ix) = 0$

$$A\mathbb{X} - (\lambda I)\mathbb{X} = \mathbb{O}$$

Now $(A - \lambda I)\mathbb{X} = \mathbb{O}$ & $\boxed{\text{this}}$ is a homogen. system.

Facts of homogeneous systems of eqn's

① has ~~trivial~~ soln $\mathbb{X} = \mathbb{O}$

② non-trivial soln's when $(A-\lambda I)$ matrix is singular (a)

these are eigen value / vectors

(b) null space of matrix → solve $\left[A - \lambda I \;\middle|\; \begin{matrix}0\\ \vdots \\ 0\end{matrix}\right]$ → non-triv. soln. → basis for $N(A-\lambda I)$

$$N(A-\lambda I) \neq \{\mathbb{O}\}$$

span of

(c) $\det(A - \lambda I) = 0$

$\boxed{\text{So}}$ find $\boxed{\text{all}}$ $\lambda, \mathbb{X}$ so that $A\mathbb{X} = \lambda \mathbb{X}$

→ Solve $\det(A - \lambda I) = 0$

Solve
$$\underbrace{\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & - & - & a_{nn}-\lambda \end{vmatrix}}_{\text{polynomial of } \lambda \;=\; p(\lambda)} = 0$$

$\equiv$ characteristic polynomial

Solve: $p(\lambda) = 0$

ex $\quad A\mathbb{X} = \lambda\mathbb{X}$? for our married/single problem

$$A = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix} \qquad \mathbb{X} = \begin{bmatrix} M \\ S \end{bmatrix} \text{ for population}$$

Solve: $\begin{vmatrix} .7-\lambda & .2 \\ .3 & .8-\lambda \end{vmatrix} = 0$

$$(.7-\lambda)(.8-\lambda) - (.3)(.2) = 0$$

$$\lambda^2 - 1.5\lambda - 0.04 = 0$$

$$\lambda = \frac{1.5 \pm \sqrt{1.5^2 + 4(0.04)}}{2} = \frac{1.5 \pm \sqrt{2.25 + .16}}{2}$$

$$\lambda = \frac{1.5 \pm \sqrt{2.41}}{2}$$

$$\lambda_1 = \frac{1.5 + \sqrt{2.41}}{2} \qquad \lambda_2 = \frac{1.5 - \sqrt{2.41}}{2}$$

$\mathbb{X}_1 = ?$ $\qquad \mathbb{X}_2 = ?$

know: $A\mathbb{X}_1 = \lambda_1\mathbb{X}_1$

means $(A - \lambda_1 I)\mathbb{X}_1 = 0$ $\to$ $\mathbb{X}_1$ is in $N(A-\lambda_1 I)$

$\to$ need basis of $N(A-\lambda_1 I)$

to find basis of $N(A-\lambda_1 I)$

Solve $\left[A - \lambda_1 I \,\middle|\, \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right]$

call basis vectors to be $\mathbb{X}_1$.

ex $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ find λ, x eigen value / vector

Step 1 λ's

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\lambda = 1 \quad \lambda = 2 \quad \lambda = 1$$

Step 2 for each λ_i you need to find basis $N(A - \lambda_i I)$

Case #1 $\lambda_1 = 2$ $\quad N\left(\begin{matrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{matrix}\right)$

to find basis $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ $x_2 = \alpha$, $x_3 = 0$ so $x_1 = \alpha$ $\quad x = \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix}$

free

$x = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\boxed{\lambda_1 = 2 \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

Case #2 $\lambda_2 = 1$ $\quad N\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$ basis?

$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = \alpha$, $x_3 = \beta$, $x_2 = -\beta$

free free

$$x = \begin{bmatrix} \alpha \\ -\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1$ x's

$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$

ex $A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad \lambda, X$ eigen value / vectors ?

Step 1 $\lambda = ?$

$$\begin{vmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$-1\begin{vmatrix} 4-\lambda & 1 \\ 1 & -1 \end{vmatrix} + (-1-\lambda)\begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$-1(\lambda-5) + (-1-\lambda)(\lambda^2-4\lambda+5) = 0$$

$$-\lambda + 5 - \lambda^3 + 3\lambda^2 - \lambda - 5 = 0$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda = 0$$

$$-\lambda(\lambda^2 - 3\lambda + 2) = 0$$

$$-\lambda(\lambda-1)(\lambda-2) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 2$$

Step 2 for each λ_i find X_i ← from basis $N(A - \lambda_i I)$

Case #1 $\lambda_1 = 0$ $\quad N\left(\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 4 & -5 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↑ free

$x_3 = \alpha$

$x_2 = \alpha$

$x_1 = \alpha$

$$X = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda_1 = 0 \quad X_1 = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$$

Case #2 $\lambda_2 = 1$

$$N\left(\begin{bmatrix} 3 & -5 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -2 \end{bmatrix}\right)$$ basis?

$$\begin{matrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{matrix}$$

$$\left[\begin{array}{ccc|c} 3 & -5 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$x_3 = \alpha$
$x_2 = 2\alpha$
$x_1 = 3\alpha$

$$\to X = \begin{bmatrix} 3\alpha \\ 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda_2 = 1, \quad X_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

Case #3 $\lambda_3 = 2$ etc..

Properties of eigen values / vectors

① if A is triangular $\lambda_i = a_{ii}$

② $|A - \lambda I| = 0$
$p(\lambda)$ has real coeff. of degree n.

(by knowing college algebra) we have n roots over the complex numbers.
and complex λ_i come in conj. pairs.
so if $\lambda_1 = a+bi$ $\lambda_2 = a-bi$
and Z_1 is λ_1's eigen vector $\bar{Z}_1 = Z_2$

ex If $\lambda_1 = 3 - 2i \qquad z_1 = \begin{bmatrix} 2+i \\ 4 \end{bmatrix}$

conj pair $\lambda_2 = 3 + 2i \qquad z_2 = \begin{bmatrix} 2-i \\ 4 \end{bmatrix}$

3) $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \cdots \lambda_n$

4) $\underbrace{\boxed{a_{11} + a_{22} + \cdots + a_{nn}}}_{= \, tr(A)} = \boxed{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$