Math 511

Q5 $\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ $\lambda_1 = 0$ $\lambda_2 = 1$ $\lambda_3 = 2$

Case 3 $\lambda_3 = 2$ $N\left(\begin{bmatrix} 2 & -5 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & -3 \end{bmatrix}\right)$ $\left[\begin{array}{ccc|c} 2 & -5 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & -5 & 1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free

$x_3 = \alpha$
$x_2 = 3\alpha$
$x_1 = 7\alpha$

$X = \begin{bmatrix} 7\alpha \\ 3\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_3 = 2$ $X_3 = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

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ex $X = \begin{bmatrix} 2+i \\ i \\ -1 \end{bmatrix} \xleftrightarrow[\text{direction}]{\text{same}} \begin{bmatrix} -1+2i \\ -1 \\ -i \end{bmatrix} = X$

6.2 / 6.3 Applications of Eigen values / vectors

6.2 Systems of Linear Diff. Equ's

system: $y_1(t), y_2(t), \ldots, y_n(t)$ amounts of y_i @ time t.

$y_1'(t), y_2'(t), \ldots, y_n'(t)$ rates of change of y_i @ time t.

system of 1st order diff eqn's

$$\begin{cases} y_1'(t) = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ y_2'(t) = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ \vdots \\ y_n'(t) = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix}$$

Solve?: $\boxed{Y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}}$

1D $\quad y' = y \to y(t) = e^t$

$y' = ay \to y(t) = ce^{at} \qquad y' = a(ce^{at}) = ay$

n-D $\quad$ Guess: $Y = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = e^{\lambda t} X$

check: $Y' = \lambda \boxed{e^{\lambda t} X} = \lambda Y$

$AY = A e^{\lambda t} X = e^{\lambda t} A X$

Soln $\quad \boxed{A} Y = Y' \qquad Y = e^{\lambda t} X \qquad$ λ is an eigen value of A; X is λ's eigen vector

So solns are $Y_1 = e^{\lambda_1 t} X_1$

$Y_2 = e^{\lambda_2 t} X_2$

$\vdots$

$Y_n = e^{\lambda_n t} X_n$

$$Y = c_1 Y_1 + c_2 Y_2 + \dots + c_n Y_n$$

or

$$Y = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \dots + c_n e^{\lambda_n t} X_n$$

arb. const.

Soln

ex

$$y_1' = 3y_1 + 2y_2$$
$$y_2' = 3y_1 - 2y_2$$

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

do 6.1 work to find $\boxed{\lambda, X}$ (examples 6.1)

$$\lambda_1 = 4 \qquad \lambda_2 = -3$$
$$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Soln: $$\boxed{Y = c_1 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

$$Y = \begin{bmatrix} c_1 2e^{4t} + c_2(-1)e^{-3t} \\ c_1 e^{4t} + c_2(3)e^{-3t} \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

6.3 we can (1) $A = LU$ ← by type 3 row ops

(2) $A = QR$ ← gram-schmidt

(3) $A = XDX^{-1}$

→ diagonalization

b/c λ_1, X_1 are eigen value/vector $AX_1 = \lambda_1 X_1$

$$
\begin{aligned}
A\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1 \\
A\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2 \\
A\mathbf{x}_3 &= \lambda_3 \mathbf{x}_3 \\
&\ \vdots \\
A\mathbf{x}_n &= \lambda_n \mathbf{x}_n
\end{aligned}
$$

call $X = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]$

then
$$
\begin{aligned}
AX &= [A\mathbf{x}_1\ A\mathbf{x}_2\ \cdots\ A\mathbf{x}_n] \\
&= [\lambda_1\mathbf{x}_1\ \lambda_2\mathbf{x}_2\ \cdots\ \lambda_n\mathbf{x}_n] \\
&= [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \lambda_n \end{bmatrix}
\end{aligned}
$$

so $X = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_n]$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}$$

$$\boxed{AX = XD}$$

$\to A = XDX^{-1}$

$\to D = X^{-1}AX$

ex $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ find $\lambda, \mathbf{x}$ (examples 6.1)

$\lambda_1 = 4$ $\quad \lambda_2 = -3$

$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$(AX = XD)$

$AX = XD$ (1) $\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$

$A = XDX^{-1}$ (2) $\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1}$

$D = X^{-1}AX$ (3) $\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

why? $\quad A\mathbf{v} \qquad X \underline{\underline{D}} X^{-1} \mathbf{v}$

(1) change from standard coord to a basis based on eigen vectors

(2) $A^K = (X D X^{-1})^K = (X D \underbrace{X^{-1})(X}_{I} D \underbrace{X^{-1})\cdots(X}_{I} D X^{-1})$

$= X D D D \cdots D X^{-1}$

$= X D^K X^{-1}$

$= X \begin{bmatrix} \lambda_1^K & & 0 \\ & \lambda_2^K & \\ 0 & & \ddots \lambda_n^K \end{bmatrix} X^{-1}$ B

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(1) Review Exam 3 12 probs @ 10 pts 110 pts = 100%

(2) Note: Final exam = Exam 1 + 2 + 3 get it to 16 probs

→ I will replace (if final % > lowest exam) lowest exam with that.

Ex

70%	50%	75%	73%
E1	E2	E3	final

→ actual grades: 70, 73, 75, 73

Exam 3 (for this weekend)

todo: ch 5 $\mathbb{R}^n$ with normal scalar product $X^T Y$

(1) do $X^T Y$

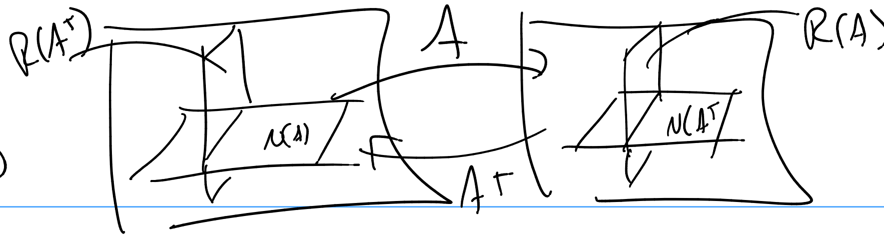
(2) understand $X^T Y$

(3) know $\cos\theta = \dfrac{X^T Y}{\|X\| \|Y\|}$

(4) $\|v\|$

(5) projections

A as a transform with (Fundamental subspaces)



Applications:

① data fit

② Inner Product spaces

③ orthonormal sets, orthogonal matrices

④ gram - schmidt

Eigen Value / Vector.