Math 511

Q's Final Exam:

Exam 1 → 5 probs
Exam 2 → 5 probs
Exam 3 → 5 probs
→ Make variations

Exam 3 Review (12 probs @ 10 pts 110 pts = 100%)

ch5 Inner Product Spaces

① (Scalar product) $\mathbb{R}^n$

$\langle x, y\rangle = x^T y$

Know: x onto y

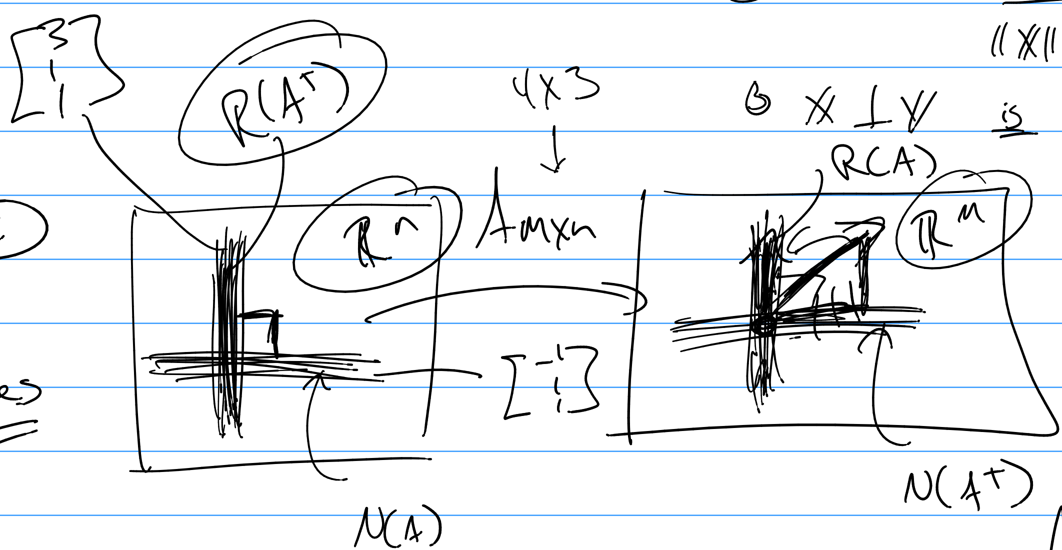
$\alpha = \dfrac{\langle x, y\rangle}{\|y\|}$

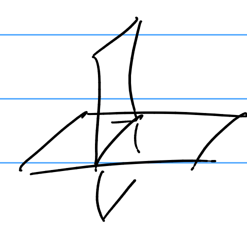
① projection x on y: $p = \alpha \dfrac{1}{\|y\|} y = \dfrac{\langle x, y\rangle}{\langle y, y\rangle} y$

② $\cos\theta = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$

③ $x \perp y$ is $\langle x, y\rangle = 0$

② Fundamental Subspaces





$N(A)$ nullspace of A
$R(A^T)$ or A's rowspace as col.

$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{b}$

(3) **Least Squares:** Data: $(x_0,y_0),(x_1,y_1),\ldots,(x_n,y_n)$

fit it to poly: ex cubic $y = a + bx + cx^2 + dx^3$

Data: (x_0,y_0) $\quad a+bx_0+cx_0^2+dx_0^3 = y_0$

(x_1,y_1) $\quad a+bx_1+cx_1^2+dx_1^3 = y_1$

$\vdots$

System

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$A\,c = \mathbb{y}$ $\leftarrow$ no soln

so $A^T A c = A^T \mathbb{y}$ $\leftarrow$ have soln it is called the least squares soln.

$\boxed{4}$ $C[-1,1]$ this is #1 except in $C[-1,1]$ and not $\mathbb{R}^n$

$\langle f,g\rangle = \int_{-1}^{1} fg\,dx$

with formulas of #1

$\mathbb{x}$ onto $\mathbb{y}$

Know

$$\alpha = \frac{\langle \mathbb{x},\mathbb{y}\rangle}{\|\mathbb{y}\|}$$

$$\|\mathbb{y}\| = \sqrt{\langle \mathbb{y},\mathbb{y}\rangle}$$

$$\mathbb{P} = \alpha\,\frac{1}{\|\mathbb{y}\|}\,\mathbb{y}$$

$\mathbb{x} = 1 \qquad \mathbb{y} = x^2$

$$\alpha = \frac{\langle \mathbb{x},\mathbb{y}\rangle}{\|\mathbb{y}\|} = \frac{2/3}{\sqrt{2/5}} = \frac{\sqrt{10}}{3}$$

$$\|\mathbb{y}\| = \sqrt{\langle \mathbb{y},\mathbb{y}\rangle} = \sqrt{\int_{-1}^{1} x^2\cdot x^2\,dx} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbb{x},\mathbb{y}\rangle = \int_{-1}^{1} 1\cdot x^2\,dx = \frac{2}{3}$$

$$\boxed{\mathbb{P} = \frac{\sqrt{10}}{3} \frac{1}{\sqrt{4/5}} x^2 = ?}$$

(5) Similar to #1, #4 except Matrix space.

to know: projection, inner products on $\mathbb{R}^n$, $\mathbb{R}^{n \times n}$, $C[a,b]$ and polynomial space P_n

n-terms or $\deg = (n-1)$

(ex) $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\boxed{\text{proj. } A \text{ onto } B}$ and $\cos\theta$ between A, B

$\mathbb{X}$ onto $\mathbb{Y}$

$\|\mathbb{Y}\| = \sqrt{\langle \mathbb{Y}, \mathbb{Y} \rangle}$

$\alpha = \dfrac{\langle \mathbb{X}, \mathbb{Y} \rangle}{\|\mathbb{Y}\|}$

$\mathbb{P} = \alpha \dfrac{1}{\|\mathbb{Y}\|} \mathbb{Y}$

$\|A\| = \sqrt{1+1+0+9} = \sqrt{11}$

$\|B\| = \sqrt{4+1+1+4} = \sqrt{10}$

$\alpha = \dfrac{\langle A, B \rangle}{\|B\|} = \dfrac{7}{\sqrt{10}}$

$$\mathbb{P} = \frac{7}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 7/5 & 7/10 \\ 7/10 & 7/5 \end{bmatrix}}$$

$$\cos\theta = \frac{\langle \mathbb{X}, \mathbb{Y} \rangle}{\|\mathbb{X}\| \|\mathbb{Y}\|} \rightarrow \boxed{\cos\theta = \frac{7}{\sqrt{11}\sqrt{10}}}$$

(6) Uses: orthonormal basis $[\mathbb{b}_1 \ \mathbb{b}_2 \cdots \mathbb{b}_n] = B$

write $\mathbb{X} = [\ \ ]_B$ $\mathbb{Y} = [\ \ ]_B$

coord. coord

$\langle \mathbb{X}, \mathbb{Y} \rangle$ = scalar prod using basis $\mathbb{b}$ coord.

orthonormal basis $\cos(2x), \sin(x) \to a\sin x + b\cos 2x = \begin{bmatrix} a \\ b \end{bmatrix}$

$f = (3\sin x - \cos(2x)) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$g = (2\cos 2x + 4\sin x) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (3\sin x - \cos 2x)(2\cos 2x + 4\sin x)\,dx = 3\cdot 4 + (-1)(2)$

$= \boxed{10}$

(7) given $p_1 = x \quad p_2 = \frac{1}{2}(3x^2-1) \quad p_3 = \frac{1}{2}(5x^3-3x)$

$\langle p, q \rangle = \int_{-1}^{1} pq\,dx$

verify orthogonal polynomials

show

$\langle p_1, p_2 \rangle = 0$

$\langle p_1, p_3 \rangle = 0$

$\langle p_2, p_3 \rangle = 0$

(8) $\boxed{\text{Gram-Schmidt}} \to A = \boxed{Q}\boxed{R}$

find find

9, 10, 12 given $A \to$ find all $\lambda_i, \vec{x}_i$

ex (12) $\begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda_1 = 9 \quad \lambda_2 = 4 \quad \lambda_3 = 1$

$\boxed{\text{case 1}}$ $\lambda_1 = 9$ $N\left(\begin{bmatrix} 0 & -5 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & -8 \end{bmatrix}\right)$ $\left[\begin{array}{ccc|c} 0 & -5 & 3 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right] \to$

$r_1 - r_2 = N r_3$

$r_2 \leftrightarrow r_3$ swap

$\left(\left[\begin{array}{ccc|c} 0 & \boxed{-5} & 3 & 0 \\ 0 & 0 & \boxed{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\right)$

free

$x_1 = \alpha$ $\quad X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \lambda_1 = 9 \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x_3 = 0$

$x_2 = 0$

case 2 case 3 <u>you do?</u>

$$X = \begin{bmatrix} 1 & X_2 & X_3 \\ 0 & & \\ 0 & & \end{bmatrix} \qquad D = \begin{array}{c} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \lambda_1 \;\; \lambda_2 \;\; \lambda_3 \end{array}$$

#11 <u>similar</u> to application #1 p-303