Math 511

Final Review:

Exam 1 → 5 probs
Exam 2 → 5 probs
Exam 3 → 5 probs
@ 10 pts each
140 pts = 100%

Final → Tuesday 1pm – 2^{50}pm

Study Ideas:

Exam 1 probs today
Exam 2 probs Sat
Exam 3 probs Sun

"Rest" Monday (do not cram)

Note:

← "on" test
~~#~~ ← not on test
? # ← study, it may be on test

Math 511 - Final Review

Exam 1

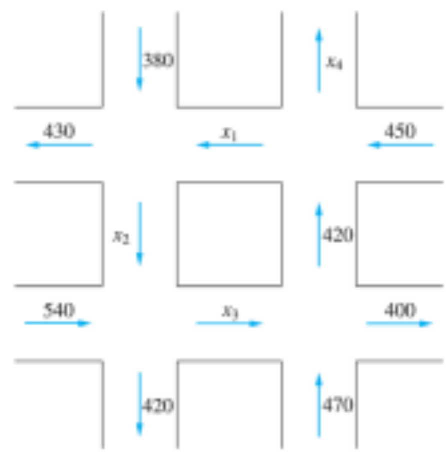
1) Solve the system of equations. DO NOT use matrices.

$$\begin{aligned} 2y + 2u &= 8 \\ x - 2y + z + u &= 0 \\ 2x + y + z - u &= 3 \\ x + y + u &= 5 \end{aligned}$$

2) Solve the system of equations. Use Gaussian Elimination on an augmented matrix.

$$\begin{aligned} 2y + 2u &= 8 \\ x - 2y + z + u &= 0 \\ 2x + y + z - u &= 3 \\ x + y + u &= 5 \end{aligned}$$

3) Determine the values of x_i for the traffic flow diagram by using Gauss-Jordan elimination on an augmented matrix.



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4) Perform the indicated operations.

a) $\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^T - 2 \begin{pmatrix} x & -y \\ y & 2x \end{pmatrix}$

b) $\begin{pmatrix} a & b & c \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

c) Calculate $I + A + A^2$ for the matrix A ...

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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0
5) Verify the given matrices are inverses.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \text{and } D = \begin{pmatrix} 0.5 & 0 & 0 \\ -1 & 1 & 0 \\ 2.5 & -2 & 1 \end{pmatrix}$$

6) Prove that if $BA = A$ and B is not the identity matrix, then A^{-1} does not exist.

7) Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}, \text{ and } X = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix}$$

Solve $AX + 2B = X + C$ for matrix C.

7 8) Find A^{-1} for the the given matrix.

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$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

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9) Find the LU factorization for the the given matrix.

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$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 2 & 4 & 4 \end{pmatrix}$$

10a) State Theorem 1.3.1

b) State Theorem 1.5.2

c) Let A be a 3 x 3 matrix, what are $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$ in relation to matrix A? If $\mathbf{a_1} + \mathbf{2a_2} = \mathbf{a_3}$, then how many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain. Is A invertable? Explain.

7 11) Given matrix A

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$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 5 & 3 & 1 \end{pmatrix}$$

a) Find $\det(A)$ by co-factors.

b) Find $\det(A)$ by elimination.

c) Does A have an inverse? Explain.

7 12) Given matrix A

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$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

What conditions must the scalars a, b, and c satisfy for A to be singular?

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0$$

$$= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$$

$$= (b-a)(c-a)[c+\cancel{a}-b-\cancel{a}]$$

$\det(A) = 0$

$D = (b-a)(c-a)(c-b)$

So $\det(A) = 0$ if $a=b$ or $b=c$ or $a=c$

EXAM 2

1) For the set of vectors in $\mathbb{R}^2$ define addition normally but scalar multiplication by $\alpha\boldsymbol{x} = [x_1, \alpha x_2]^T$. Does this form a vector space? Explain. (Note: Axioms are given on the last page of the exam)

$X + Y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ $\quad \alpha X = \begin{bmatrix} x_1 \\ \alpha x_2 \end{bmatrix}$

Know $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $C[a,b]$, P_n, P

2) Does the set of all 2 x 2 matrices A such that $a_{22} = 1$ form a subspace of $\Re^{2\times 2}$? Explain.

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

3) Let $\boldsymbol{x}_1 = [1, 0, -1]^T, \boldsymbol{x}_2 = [2, 4, 0]^T$, and $\boldsymbol{x}_3 = [0, 2, 1]^T$. Are the vectors linearly independent? Prove your answer.

a) Square use det()

4) Are $\{(1), (1 + x), (x + x^2), (x^2 + 1)\}$ linearly independent in P_3? Prove your answer.

b) can't use det() overdetermined 3x5 → row ops to find lead/free var

5) Consider the vectors $\boldsymbol{x}_1 = [1, 2, 1]^T, \boldsymbol{x}_2 = [2, 5, 0]^T, \boldsymbol{x}_3 = [1, 3, -1]^T$, and $\boldsymbol{x}_4 = [3, 7, 1]^T$. What is the dimension for the Span of the vectors? Pare down and/or extend the vectors to make a basis for $\Re^3$.

c) linear combo = zero (solve → only soln = 0 → non-all 0 soln)

6) Consider the polynomials $(1 - x)$ and $(1 + x^2)$. Extend the polynomials to make a basis for P_3.

$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

7) For R^2 with bases $B_1 = \{[1, -1]^T, [1, 2]^T\}$ and $B_2 = \{[1, -3]^T, [2, 7]^T\}$, write the transition matrix, and call it S, representing the change of base from B_2 to B_1. Write the transition matrix, and call it T, representing the change of base from B_1 to B_2. DO NOT MULTIPLY THE MATRICES OR FIND ANY INVERSES FOR YOUR ANSWERS. Just leave them as a product of matrices and their inverses.

this will be also part of a Exam 3 prob.

8) Let A be a 3×4 matrix and U is the reduced row echelon form of A. If ...

$$U = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$u_{14} = u_{11} + 2u_{13}$

$\boxed{a_{14} = a_{11} + 2a_{13} \quad a_{12} = -2a_{11}}$

a) determine the rank(A), the nullity of A, and the dependency equations.

b) if $\boldsymbol{a}_1 = [1, 2, 3]^T, \boldsymbol{a}_2 = [-2, -4, -6]^T, \boldsymbol{a}_3 = [-1, 1, -2]^T$, and $\boldsymbol{a}_4 = [-1, 4, -1]^T$ then write the basis vectors for the column space of A.

c) write the basis vectors for the row space of A.

d) write the basis vectors for N(A).

9) Determine if $L([x_1, x_2]^T) = [x_1, x_1, x_1 + x_2^2]^T$ from $\Re^2$ to $\Re^3$ is a linear operator.

$P_3 = a + bx + cx^2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

~~10~~) Determine the kernel and range of the linear operator $L(\boldsymbol{p}) = x\boldsymbol{p}' + \boldsymbol{p}''$ on P_3.

$L\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c \\ b \\ 2c \end{bmatrix} =$ $L(p) = \begin{bmatrix} 2c \\ b \\ 2c \end{bmatrix}$

$p' = \frac{d}{dx}[a + bx + cx^2] = b + 2cx = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$

11) For the linear operator $L(\boldsymbol{x}) = [x_1, x_2, x_1 + 2x_2]^T$ from $\Re^2$ into $\Re^3$ find the standard linear operator matrix, A_E.

one prob.

$p'' = \frac{d}{dx}[b + 2cx] = \begin{bmatrix} 2c \\ 0 \\ 0 \end{bmatrix}$

12) For the linear operator $L(\boldsymbol{p}) = x\boldsymbol{p}' + \boldsymbol{p}''$ on P_3 ...

... with standard basis $E = [(1), (x), (x^2)]$ and basis $B = [(1), (1 + x), (x + x^2)]$ $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

a) Find the matrix representation of L with respect to the standard basis, and call it A_E. DO NOT MULTIPLY THE MATRICES OR FIND ANY INVERSES FOR YOUR ANSWERS. Just leave them as a product of matrices and their inverses.

b) Find the matrix representation of L with respect to basis B, and call it A_B. DO NOT MULTIPLY THE MATRICES

$L\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c \\ 6 \\ 2c \end{bmatrix}$

OR FIND ANY INVERSES FOR YOUR ANSWERS. Just leave them as a product of matrices and their inverses.

$A_E = L(\text{standard basis}) = [\,L(1),\ L(x),\ L(x^2)\,] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Exam 3

1) For the pair of vectors $\boldsymbol{x} = (1,2,3)^T$ and $\boldsymbol{y} = (1,1,1)^T$, using the scalar product find the scalar projection α of $\boldsymbol{x}$ onto $\boldsymbol{y}$ and the vector projection $\boldsymbol{p}$ of $\boldsymbol{x}$ onto $\boldsymbol{y}$.

but I pick inner product space: (See #1, #4, #5, #7)

(See below)

3x4 if I used #8 & Exam 2

2) Let A be a ~~3x5~~ matrix. Considering A as a linear transform describe its domain and codomain and draw a visual example as has been done in class. For the codomain, is it possible for A to have the vector $(2,1,2)^T$ in its column space and $(-1,1,1)^T$ in the null space of A^T? Explain.

→ This problem and #8 Exam 2 combined

7, 6

3) For an experiment you collect the following (x,y)-data points: $\{(1,1),(2,1),(3,2),(4,2),(5,1)\}$. Setup the matrices and equation to solve the least-squares fit to the data by a cubic polynomial. DO NOT solve the system, but explain the steps you would take to solve it.

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4) Given inner product space $\mathbb{R}^{2\times 2}$ with $\langle A,B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

find the projection of A onto B.

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5) In $C[-1,1]$, with inner product $< f,g >= \int_{-1}^{1} f(x)g(x)dx$, knowing that for an inner product space $\cos(\theta) = \frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$, find the angle between $f(x) = 1$ and $g(x) = x^2$. (Note: leave your answer in arccos form)

7, 0

6) The functions $\cos(x)$ and $\sin(x)$ form an orthonormal set in $C[-\pi,\pi]$ with the inner product defined by $< f,g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$. Determine the value of ...

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{(4\sin(x) - 2\cos(x))}_{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}\underbrace{(\cos(x) + 3\sin(x))}_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}dx = 12 - 2 = \boxed{10}$$

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7) With respect to the inner product $< f,g >= \int_{-1}^{1} f(x)g(x)dx$ are the Legendre Polynomials $p_1(x) = x$, $p_2(x) = \frac{1}{2}(3x^2 - 1)$, and $p_3(x) = \frac{1}{2}(5x^3 - 3x)$ orthogonal?

8) Use the Gram-Schmidt process to find the QR factorization of $A = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}$

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9) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -3 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{pmatrix} = [\ X\ ][\ D\ ][\ X\ ]^{-1}$$

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10) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$$

11) ~~Two tanks each contain 200 liters of a mixture. Initially, the mixture in tank A contains 60 grams of salt while tank B contains no salt. Pure water is pumped into tank A at 15 L/min, the mixture from tank B is pumped into tank A at 5 L/min, the mixture from tank A is pumped into tank B at 20 L/min, and the mixture from tank B is pumped out of the system at 15 L/min. Draw a figure representing this system. Setup the inital value problem of the form $\boldsymbol{Y}' = A\boldsymbol{Y}$, $\boldsymbol{Y}(0) = \boldsymbol{Y}_0$. DO NOT solve.~~

eigen problem (#9, #10)

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12) For the given matrix A, find the equation $\boxed{AX = XD}$ where D is a diagonal matrix.

$$A = \begin{pmatrix} 2 & -5 \\ 0 & -3 \end{pmatrix}$$

$A = [\quad]$ give you $U = [\quad]$

3x4

Find, subspaces