Math 511

$\boxed{Q's}$ 1.4 #4

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $A^n = 0$ f $n \geq 4$

$A^2 = AA$

$A^3 = A^2 A$

$A^4 = A^3 A$

$A^5 = A^4 A$

$A^6 = A^4 A^2$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{} = \overbrace{\boxed{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}}^{A^2}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{etc}$$

1.4 #12

$d = a_{11}a_{22} - a_{21}a_{12}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \frac{a_{22}}{d} & \frac{-a_{12}}{d} \\ \frac{-a_{21}}{d} & \frac{a_{11}}{d} \end{bmatrix} \stackrel{?}{=} I$$

$$\begin{bmatrix} \frac{a_{22}}{d} & \frac{-a_{12}}{d} \\ \frac{-a_{21}}{d} & \frac{a_{11}}{d} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \stackrel{?}{=} I$$

#14 f $AB = A$ and $B \neq I$ $\to$ A is singular

(A is not invertible)

(A^{-1} does not exist)

Pf: well if A^{-1} exists.

$$AB = A$$
$$A^{-1}AB = A^{-1}A$$
$$IB = I$$
$$B = I$$

but $B \neq I$ so my start (A^{-1} exists) is wrong. (A^{-1} doesn't exist)

$(I-A)^{-1}$ exists

#16 if $A^2 = O$ then $(I-A)$ is non-singular

and $(I-A)^{-1} = I + A$

check $(I-A)(I+A) = II + IA - AI - AA$

$= I + A - A - A^2$

$= I + O - O$

$= I$

$(I+A)(I-A) \stackrel{?}{=} = I$

Find A^{-1} $\quad A_{n\times n}$ $\quad [A \mid I] \xrightarrow[\text{ops}]{\text{row}} [I \mid A^{-1}]$

Note: if you would like to find $A^{-1}B$

ex $\quad AC = B$

$A^{-1}AC = A^{-1}B$

$C = A^{-1}B$

you can use

$[A \mid B]$

row ops

$[I \mid A^{-1}B]$

When does A^{-1} exist?

$[A] \xrightarrow[\text{ops}]{\text{row}} [I]$

ex $A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 3 & -1 \\ 2 & 1 & -5 \end{bmatrix}$ A^{-1} exist?

Notice $(2)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 3 & -1 \\ 2 & 1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{Q^2}$ find a way to determine if A^{-1} exists.

($\underset{n\times n}{A}$ is supposed to be row-equiv. to I)

ex $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{(-c)} \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

[1x1] $A = [a]$ $[a \mid 1] \to [1 \mid 1/a]$ $a \neq 0$

$\underline{1\times 1}$ $A = [a]$ $a \neq 0$ for A^{-1} to exist

$\underline{2\times 2}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $ad-bc \neq 0$ for A^{-1} to exist

3x3 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Def $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & - & - & a_{nn} \end{bmatrix}$

$n \times n$

① $\det(A)$ is a formula such that

if $\det(A) = 0$ A^{-1} does not exist, A is singular

$\det(A) \neq 0$ A^{-1} exists, A is not-singular

② M_{ij} is a $(n-1) \times (n-1)$ matrix made by removing row i col j of A.

$\det(M_{ij})$ is called a_{ij} minor

③ $A_{ij} = (-1)^{i+j} \det(M_{ij})$ cofactor of a_{ij}

So $\det(A)$ by co-factor expansion.

① pick row or col. (ex. row k)

$A = \begin{bmatrix} & & & \\ \boxed{a_{k1} & a_{k2} & \cdots & a_{kn}} \\ & & & \end{bmatrix}$

$\det(A) = a_{k1} A_{k1} + a_{k2} A_{k2} + \cdots + a_{kn} A_{kn}$

start

2×2 $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

(a) $\det\left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right) = 2+3 = 5 \neq 0$ so A^{-1} exists

$\det\left(\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\right) = -2+2 = 0$ so A^{-1} does not exist.

(ex) $\det\left(\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 1 & 0 & 20 \end{bmatrix}\right) = -(2)\begin{vmatrix} -1 & 1 \\ 1 & 20 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 20 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}$

$= -2(-21) + 1(17) - 0$

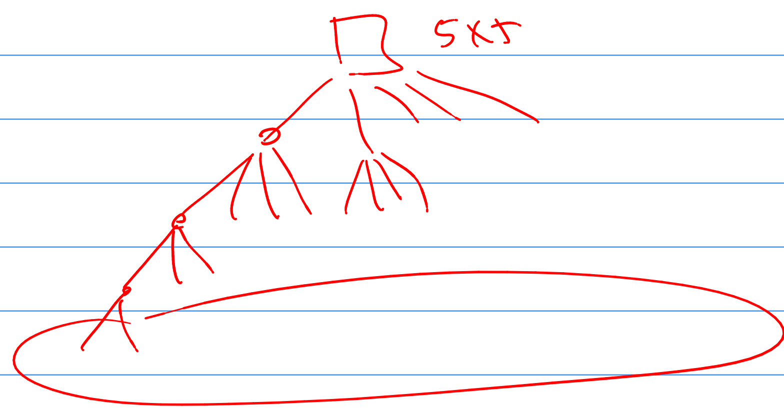
$= 42 + 17$

$= 59$

$|6\times6| = \underbrace{|5\times5| + |5\times5| + |5\times5| + \cdots + |5\times5|}_{6}$

$5\,|4\times4| \qquad 5\,|4\times4|$

5x5



p.99 10x10 matrix. Cofactor need 6,235,300 mult.

det(A) in a faster way?

use a simple example: A is triangular

ex) upper triangular

$$\det(A) = \begin{vmatrix} a_{11} & & \sim \\ 0 & a_{22} & & \sim \\ 0 & 0 & \ddots & \\ \vdots & \vdots & 0 & \\ 0 & & & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & & \sim \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & & a_{nn} \end{vmatrix}$$

$$= \boxed{a_{11} a_{22} \cdots a_{nn}}$$

ex)

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{vmatrix} = -3$$

$$E_k \cdots E_2 E_1 A = U$$