Math 511

are objects of V

Q's 3.1 #3 V set of objects with $X + Y$, αX

complex numbers with typical complex + complex, α complex

defn: $X =$

Vector Space?

(1)
(2) check closure.

Def: for an operator to be "closed" means
object 1 $\odot$ object 2
operator
= same type of object.

A1) check commutative.

A2) check assoc.

$$(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$$

let $c_1 = a+bi$ $c_2 = c+di$ $c_3 = e+fi$

left side: $\left((a+c) + (b+d)i \right) + (e + f i)$

$$= ((a+c)+e) + ((b+d)+f)\, i$$

right side: show: $c_1 + (c_2 + c_3)$ is same

"vector" = complex number

A1) $c_1 + c_2 = c_2 + c_1$ show

3.3 given $V_1, V_2, \ldots, V_n$ in V a vector space

collection of objects with $X + Y$, αX defined we call $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_n$ linearly independent if

#1

$$\boxed{c_1 \mathbb{V}_1 + c_2 \mathbb{V}_2 + \ldots + c_k \mathbb{V}_k = \mathbb{O}}$$

has only trivial soln (call $\mathbb{V}_i$ linearly ind.)

#2 if we have a non-trivial solution then the $\mathbb{V}_i$ are called linearly dependent.

Ex are $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$ linearly indp.

($\mathbb{R}^{2\times 2}$ Vector space)

check $c_1 \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} (c_1 + 2c_3) & (c_1 - c_2 + c_3) \\ (2c_1 + c_2 + 5c_3) & (2c_2 + 2c_3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} c_1 + 2c_3 = 0 \\ c_1 - c_2 + c_3 = 0 \\ 2c_1 + c_2 + 5c_3 = 0 \\ 2c_2 + 2c_3 = 0 \end{array} \right. \longrightarrow \text{Solve}$$

if you get $c_1 = 0$ $c_2 = 0$ $c_3 = 0$

→ Matrices are indep.

if you get $\underbrace{c_1 = [\ ], c_2 = [\ ], c_3 = [\ ]}_{\text{not all zero}}$

→ Matrices are dep.

ex) objects of P_3 $\quad p_1 = 2 + x \quad p_2 = x - x^2 \quad p_3 = 4 + x^2$

$p_4 = 4$

Lin. Ind?

check: $c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 = 0$

$$c_1(2 + x + 0x^2) + c_2(0 + x - x^2) + c_3(4 + 0x + x^2) + c_4(4 + 0x + 0x^2) = 0 + 0x + 0x^2$$

$$\begin{aligned} x^2 &: \quad -c_2 + c_3 = 0 \\ x &: \quad c_1 + c_2 = 0 \\ \text{const} &: \quad 2c_1 + 4c_3 + 4c_4 = 0 \end{aligned}$$

Solve!

$C[a,b]$ objects = cont. functions over $[a,b]$

$f_1 = \sin(x) \quad f_2 = e^x \quad f_3 = x + x^2$ over $[-1,1]$

Lin. Ind.? $\quad c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$$\begin{bmatrix} c_1 \sin x + c_2 e^x + c_3(x + x^2) = 0 \\ c_1 \cos x + c_2 e^x + c_3(1 + 2x) = 0 \\ c_1(-\sin x) + c_2 e^x + c_3(2) = 0 \end{bmatrix}$$

Solve?

$$\begin{bmatrix} \sin x & e^x & x + x^2 \\ \cos x & e^x & 1 + 2x \\ -\sin x & e^x & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

use $\det\left(\quad \right) = 0$

$\neq 0 \rightarrow f_i$ are lin. ind.

x is any number between $[a,b]$

$$\begin{vmatrix} \sin x & e^x & x+x^2 \\ \cos x & e^x & 1+2x \\ -\sin x & e^x & 2 \end{vmatrix} = \text{will be a function of } x = W_{v_i}(x)$$

if for any $x \in [a,b]$ $W_{v_i}(x) \neq 0$ then linearly ind.

if $W_{v_i}(x) = 0$ (identically)

you learned nothing!

Diff Eq.

$S_1(x) = 1 \quad S_2(x) = x \quad \cancel{S_3(x) = 1+x}$

$$S(x) = a(1) + b(x)$$

are $S_1 = 1$ $S_2 = x$ lin. ind?

$$W_{v_i}(x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \quad \text{lin. ind.!}$$

other tech. for checking for lin. ind in $C[a,b]$

$f_1, f_2, \ldots, f_k$

you tried

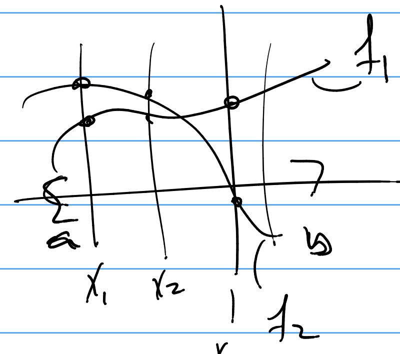
$$\begin{vmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & & f_k' \\ \vdots & & & \\ f_1^{(k-1)} & \cdots & & f_k^{(k-1)} \end{vmatrix} = \text{algebra} = 0$$

find identically zero

so wronskian fails (can't use it)

tech #2 Sample (pick x's) between [a, b]

$f_1(x), f_2(x), \ldots, f_k(x)$



$$\left| c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0 \right|$$

pick x_1: $c_1 f_1(x_1) + c_2 f_2(x_1) + \cdots + c_k f_k(x_1) = 0$

pick x_2: $c_1 f_1(x_2) + c_2 f_2(x_2) + \cdots + c_k f_k(x_2) = 0$

$\vdots$

pick x_k: $c_1 f_1(x_k) + \cdots + c_k f_k(x_k) = 0$

use

$$\begin{vmatrix} f_1(x_1) & \cdots & f_k(x_1) \\ \vdots & & \\ f_1(x_k) & & f_k(x_k) \end{vmatrix} \neq 0$$

if not zero the f_i are lin. indep.!