Math 511

Q's 3.2 #3 Vector Space is $\mathbb{R}^{2\times2}$

(a) $S = \{$ all 2×2 singular matrices $\}$

check: #1 is $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ in S $\boxed{\text{is}}$ singular

(b/c it is not row equiv to I

or $\det\left(\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)=0$)

#2 $A\in S$ and $B\in S \Rightarrow$ is $A+B\in S$

ex $\begin{bmatrix}1&0\\0&0\end{bmatrix}=A$ (know b/c $\det(A)=0$, A is sing.)

$\begin{bmatrix}0&0\\0&1\end{bmatrix}=B$ (know b/c $\det(B)=0$, B is sing.)

but $A+B=\begin{bmatrix}1&0\\0&1\end{bmatrix} \leftarrow$ is non-sing.

S is not a subspace.

#4 4c $\left[\begin{array}{ccc|c}1&3&-4&0\\2&-1&-1&0\\-1&-3&4&0\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&3&-4&0\\0&-7&7&0\\0&0&0&0\end{array}\right]$

$\rightarrow\left[\begin{array}{ccc|c}1&3&-4&0\\0&\boxed{1\;\;-1}&&0\\0&0&0&0\end{array}\right]$ $X=\begin{bmatrix}\alpha\\\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$

free $x_3=\alpha$

$x_2=\alpha$

$x_1=\alpha$

So any thing that looks like $\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$, for any α

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} ☺ \\ ☺ \\ ☺ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$N(A) = \left\{ \mathbb{v} \;\middle|\; \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbb{v},\ \alpha \in \mathbb{R} \right\}$$

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ex $\quad A = \begin{bmatrix} & & \end{bmatrix} \rightarrow \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$

row ops $\rightarrow$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = 2\beta - 2\alpha$

$x_3 = -2\beta$

$x_5 = 0$

free $x_2 = \alpha$ $\qquad$ free $x_4 = \beta$

$$\mathbb{x} = \begin{bmatrix} 2\beta - 2\alpha \\ 0\beta + 1\alpha \\ -2\beta + 0\alpha \\ 1\beta + 0\alpha \\ 0\beta + 0\alpha \end{bmatrix} = \beta \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$N(A) = \text{Span}\left([2\ 0\ -2\ 1\ 0]^T, [-2\ 1\ 0\ 0\ 0]^T \right)$$

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using $N(A)$ vectors

Fact: every $\mathbb{v} \in N(A)$ $\quad A\mathbb{v} = \mathbb{0}$

so $A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$

from above

if $\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \\ -15 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \\ -15 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

if $\quad V \in N(A) \qquad A(X+V) = AX$

3.4 **Basis / Dimensions** (use linear ind.)

Def $v_1, v_2, \dots, v_n$ are a basis for vector space V

if and only if ① $v_1, v_2, \dots, v_n$ are linearly ind.

② $\text{Span}(v_1, v_2, \dots, v_n) = V$

Standard Basis ① $\mathbb{R}^n \qquad e_1, e_2, \dots, e_n \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ on i^{th} row

(ex) $\mathbb{R}^3 \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

a) lin. ind. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ so independent

b) span? $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\mathbb{R}^{2\times 2}$ <u>standard basis</u> $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$

P_3 <u>standard basis:</u> $p_1 = 1$, $p_2 = x$, $p_3 = x^2$

ex $a + bx + cx^2$ any poly. of P_3

Th^m given {any} spanning set of n - vectors

any other collection of $m > n$ vectors are lin. dependent.

<u>Corollary:</u> given two basis sets of m-vectors and n-vectors <u>then</u> $m = n$.

Def dimension of V, $\dim(V)$, is the number of vectors in a basis for V.

<u>Facts:</u>

① $\dim(\{0\}) = 0$

② $\dim(V) = n$ is a finite number
call V to be finite dimensional

③ if $\dim(V)$ is not finite
call V to be infinite dimensional

ex $\dim(P_4) = 4$ standard basis

$p_1 = 1$, $p_2 = x$, $p_3 = x^2$, $p_4 = x^3$

ex Vector space P of poly. of $\boxed{\text{any}}$ length.

Fact: $\dim(P)$ = not finite (infinite)

pf: if $\dim(P)$ is finite. say $\dim(P) = n$

(know any more poly. must be dep.)

$p_1 = 1,\ p_2 = x,\ p_3 = x^2, \ldots,\ p_{n+1} = x^n$

check for dep/ind. (Wronskian)

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 3\cdot 2x & \cdots & n(n-1)x^{n-2} \\ 0 & 0 & 0 & 3\cdot 2 & & n(n-1)(n-2)x^{n-3} \\ \vdots & & & & \ddots & \\ 0 & & & & & n! \end{vmatrix} = 1\cdot 1\cdot 2!\cdot 3!\cdot 4!\cdots n! \neq 0$$

ind.

So contradiction and $\dim(P)$ = infinite.

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thm if $\dim(V) = n > 0$ then

1. any set of n lin. ind. vectors span V.
2. any n vectors that span are lin. ind.
3. no set fewer than n span V.
4. any subset fewer than n-vectors that are lin. ind. can be extended (add new lin. ind. vectors) to make a basis.

⑤ given a spanning set of $m > n$ vectors,
you can pare down (remove dep. vectors)
to get a basis.