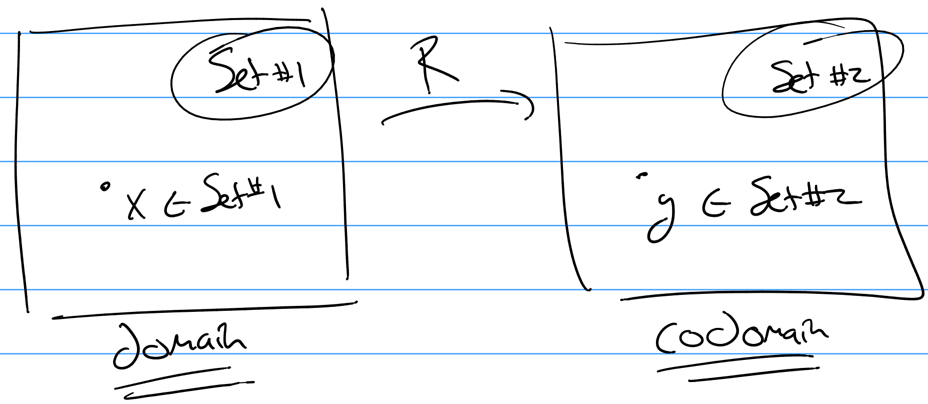
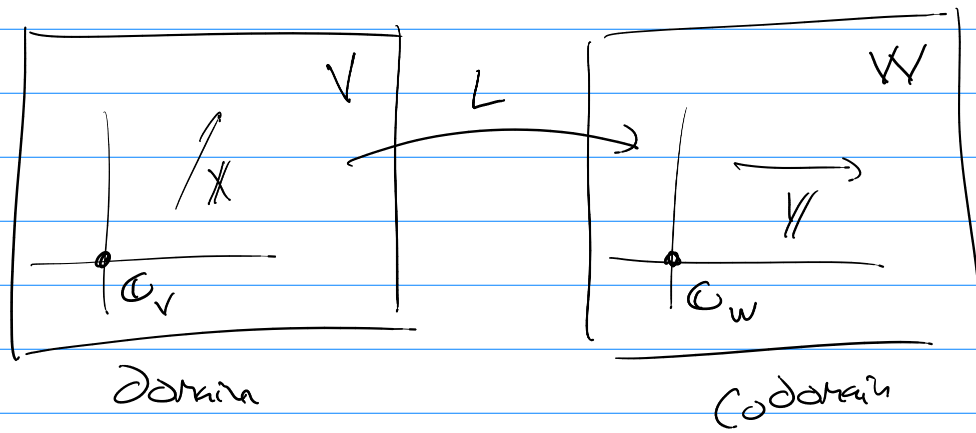
Math 511

Mappings: / Relationships (Create relationship between objects of two sets)



$R(x) = y$

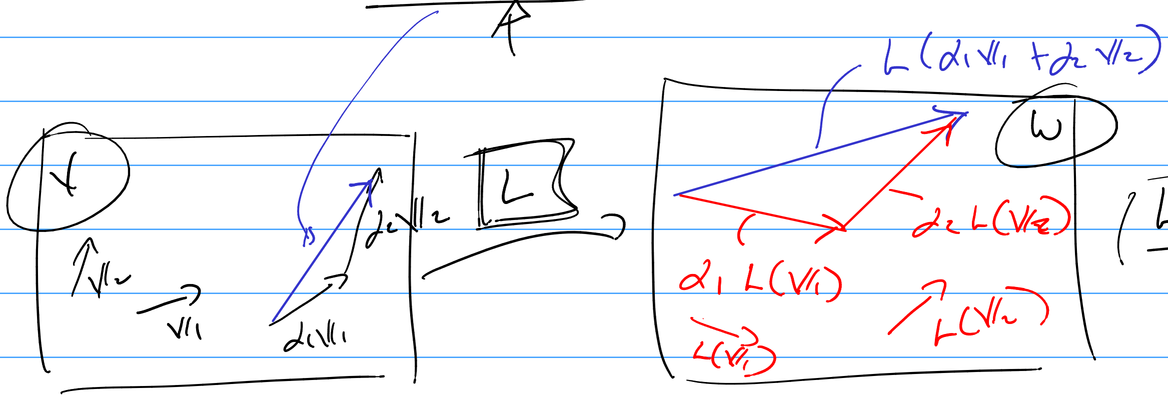
Chapter 4 **Linear Transformations** from vectorspace V to vectorspace W



$$L(\mathbb{x}) = \mathbb{y}$$

call L a linear transform from V to W if

① $L(\underline{\boxed{\alpha_1 \mathbb{v}_1 + \alpha_2 \mathbb{v}_2}}) = \alpha_1 L(\mathbb{v}_1) + \alpha_2 L(\mathbb{v}_2)$

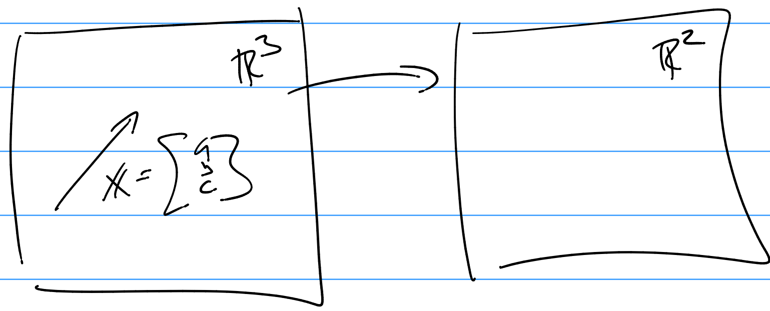


Notation:

$\boxed{L : V \to W}$

(Ex) is the given transform a linear transformation?

(ex #1)



$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} c \\ a+b \end{bmatrix}$ example: $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

book notation: $L([x_1\ x_2\ x_3]^T) = [x_3,\ x_1 + x_2]^T$

$v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $v_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

does this preserve: $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

left side: $\alpha_1 v_1 + \alpha_2 v_2 = \begin{bmatrix} \alpha_1 a + \alpha_2 d \\ \alpha_1 b + \alpha_2 e \\ \alpha_1 c + \alpha_2 f \end{bmatrix}$

now: $L(\alpha_1 v_1 + \alpha_2 v_2) = \begin{bmatrix} \alpha_1 c + \alpha_2 f \\ (\alpha_1 a + \alpha_2 d) + (\alpha_1 b + \alpha_2 e) \end{bmatrix} =$

right side: $\alpha_1 L(v_1) + \alpha_2 L(v_2) = \alpha_1 \begin{bmatrix} c \\ a+b \end{bmatrix} + \alpha_2 \begin{bmatrix} f \\ d+e \end{bmatrix} =$

$= \begin{bmatrix} \alpha_1 c + \alpha_2 f \\ \alpha_1 a + \alpha_1 b + \alpha_2 d + \alpha_2 e \end{bmatrix} =$ Same!

$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}?\\1\end{bmatrix}$ $\quad L\left(\begin{bmatrix}0\\-4\end{bmatrix}\right) = \begin{bmatrix}-4\\1\end{bmatrix}$ $\quad L: \mathbb{R}^2 \to \mathbb{R}^2$?

ex #2 $\quad L\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}a+b\\1\end{bmatrix}$ $\quad$ linear transform?

check: $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

$v_1 = \begin{bmatrix}a\\b\end{bmatrix} \quad v_2 = \begin{bmatrix}c\\d\end{bmatrix}$

left side: $L\left(\alpha_1 \begin{bmatrix}a\\b\end{bmatrix} + \alpha_2 \begin{bmatrix}c\\d\end{bmatrix}\right) = L\left(\begin{bmatrix}\alpha_1 a + \alpha_2 c\\ \alpha_1 b + \alpha_2 d\end{bmatrix}\right)$

$= \begin{bmatrix}\alpha_1 a + \alpha_2 c + \alpha_1 b + \alpha_2 d\\ 1\end{bmatrix}$

not equal!

right side $\quad \alpha_1 L(v_1) + \alpha_2 L(v_2)$

$= \alpha_1 \begin{bmatrix}a+b\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}c+d\\1\end{bmatrix} = \begin{bmatrix}\alpha_1 a + \alpha_1 b + \alpha_2 c + \alpha_2 d\\ \alpha_1 + \alpha_2\end{bmatrix}$

not a linear transform

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ex $\quad$ is derivative of $p(x)$ a polynomial a linear transform?

$P_4 \xrightarrow{D} P_4$

ex $\quad D(2 + x - x^3) = 1 - 3x^2$

$D(4) = 0$

all terms

$2 + x + 0x^2 - x^3 \xrightarrow{D} 1 + 0x - 3x^2 + 0x^3$

$\boxed{4} + 0x + 0x^2 + 0x^3 \xrightarrow{D} 0 + 0x + 0x^2 + 0x^3$

check linear transform?

$\hookrightarrow$ check $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

of P_4 $\quad$ $\underline{\underline{\mathbb{V}_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3}}$ $\qquad$ $\mathbb{V}_2 = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$\boxed{\text{left side:}}$ $L(\alpha_1 \mathbb{V}_1 + \alpha_2 \mathbb{V}_2)$

$$= L\left((\alpha_1 a_0 + \alpha_2 b_0) + (\alpha_1 a_1 + \alpha_2 b_1)x + (\alpha_1 a_2 + \alpha_2 b_2)x^2 + (\alpha_1 a_3 + \alpha_2 b_3)x^3\right)$$

$$= (\alpha_1 a_1 + \alpha_2 b_1) + 2(\alpha_1 a_2 + \alpha_2 b_2)x + 3(\alpha_1 a_3 + \alpha_2 b_3)x^2 + 0x^3$$

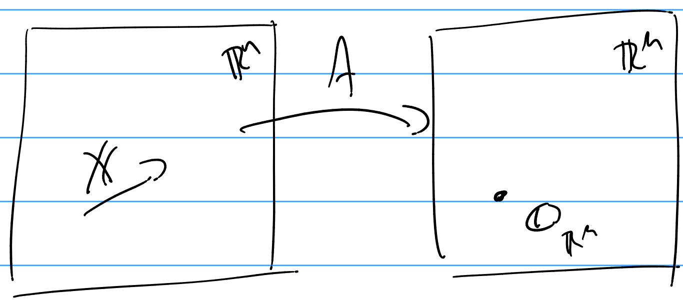
$\boxed{\text{right side}}$ $\alpha_1 \boxed{L(\mathbb{V}_1)} + \alpha_2 (\mathbb{V}_2)$ $\qquad$ $\updownarrow$ $\underline{\underline{\text{Same!}}}$

= do the calculus and algebra = $\boxed{}$

So $D : P_4 \to P_4$ $\boxed{\text{is}}$ a linear transform

$\underline{\underline{\text{Review}\quad A\mathbb{x} = \mathbb{y}}}$

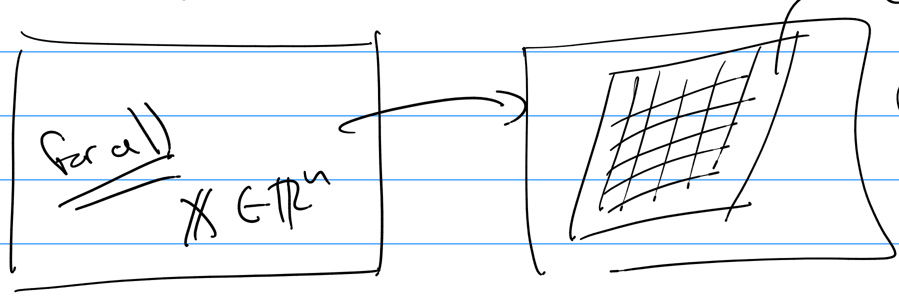
#1



$$N(A) = \{\, \mathbb{x} \mid A\mathbb{x} = 0_{\mathbb{R}^n} \,\}$$

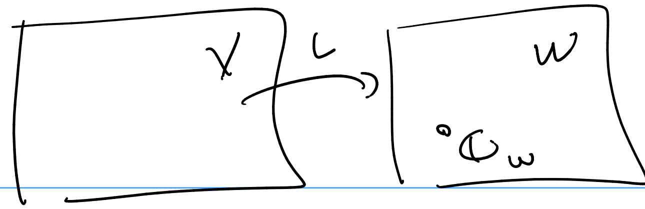
#2 $\quad$ Col space of A = Span({ $\underline{A\text{'s}}$ $\underline{\text{ind.}}$ col's })

range of A



col space of A
"
range of A
subspace of $\mathbb{R}^n$

for $L: V \to W$



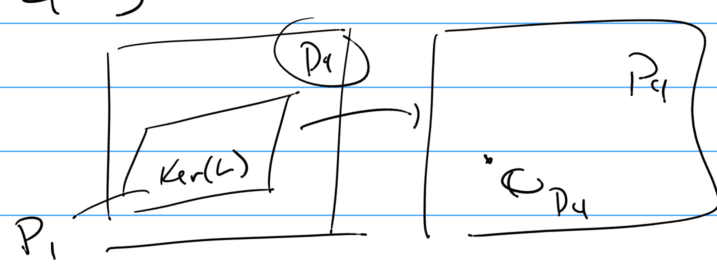
#1 Kernel of L = $\text{Ker}(L) = \{ v \in V \mid L(v) = 0_W \}$

ex $D: P_4 \to P_4$

b/c $D[a + 0x + 0x^2 + 0x^3] = 0 + 0x + 0x^2 + 0x^3 = 0_{P_4}$

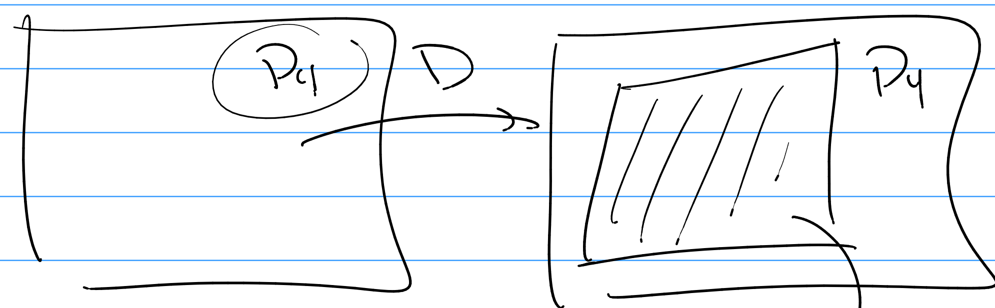
$p(x) = \text{const}$ is Kernel of D

$\text{Ker}(L) = \{ p \mid p(x) = a \}$
$= P_1$



#2 range of L = $\{ w \in W \mid$ ~~some~~ $v \in V$ exists such that
$L(v) = w \}$

ex



range of $D = P_3$

Note: every matrix $A_{m \times n}$ is a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^m$

$\boxed{Q}$ given a linear transform...

ex $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} c \\ a+b \end{bmatrix}$ $\quad L: \mathbb{R}^3 \to \mathbb{R}^2$

Can I find a matrix that does this transform?

$L(\mathbb{x}) = \mathbb{y}$ same as $A\mathbb{x} = \mathbb{y}$