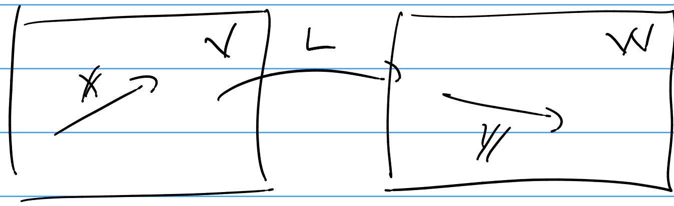
# Math 511

ch 4

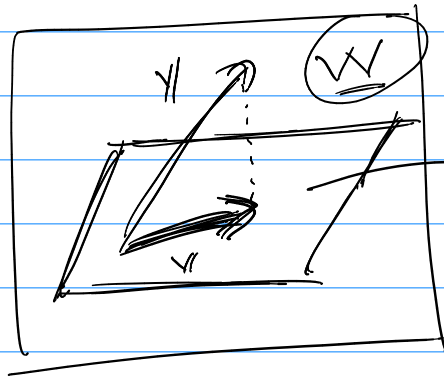



$L(\mathbf{x}) = \mathbf{y}$

$A[\mathbf{x}]_E = [\mathbf{y}]_E$

$S[\mathbf{x}]_B = [\mathbf{y}]_D$

goal



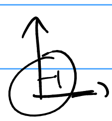

$\dim(W) = n$

Span$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$

$\alpha_1 \boxed{\mathbf{v}_1} + \alpha_2 \boxed{\mathbf{v}_2} + \cdots + \alpha_k \boxed{\mathbf{v}_k} = \mathbf{v}$

$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}$ — coord. of $\mathbf{v}$ for vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

$\mathbb{R}^2, \mathbb{R}^3$



$\boxed{5.1}$ orthogonal?

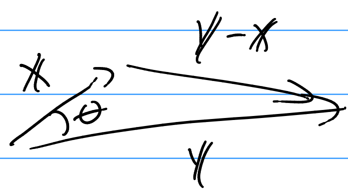
use $\mathbb{R}^2, \mathbb{R}^3$ space b/c

orthogonal

, right angles

, $90^\circ$ or $\pi/2 \pm n\pi$

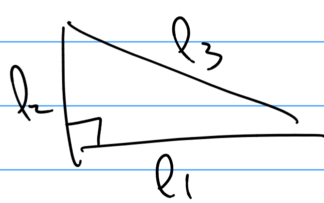
, $\cos(\pi/2) = 0$

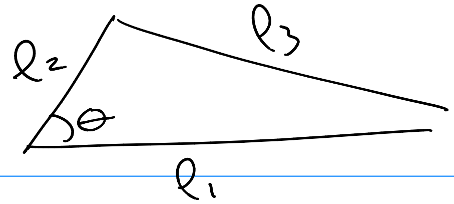



we can use law of cosines

$l_1^2 + l_2^2 = l_3^2$

trig:






$$l_3^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos\theta$$

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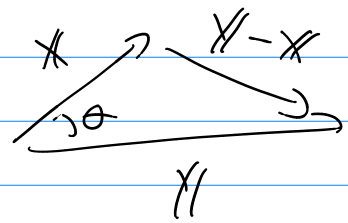
but we have vectors (need lengths)




$\mathbb{R}^2 \quad \|x\|^2 = (x_1^2 + x_2^2) = x^T x$

$\mathbb{R}^3 \quad \|x\|^2 = (x_1^2 + x_2^2 + x_3^2) = x^T x$

where: scaler product $\quad x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n$

and we know $x^T y = y^T x$

$$\|y - x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\| \cos\theta$$

$$\begin{aligned}
2\|x\|\,\|y\|\cos\theta &= \|x\|^2 + \|y\|^2 - \|(y - x)\|^2 \\
&= x^T x + y^T y - (y - x)^T (y - x) \\
&= x^T x + y^T y - y^T y - x^T x + 2 x^T y \\
&= 2 x^T y
\end{aligned}$$

$$\|x\|\,\|y\|\cos\theta = x^T y$$

$$\cos\theta = \frac{x^T y}{\|x\|\,\|y\|}$$

know $x \perp y$ if $\cos(\theta) = 0$

So only when $x^T y = 0$

if $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \rightarrow$ call $x \perp y$

**Def** $\mathbf{x}, \mathbf{y}$ are orthogonal $(\mathbf{x} \perp \mathbf{y})$
if and only if $\mathbf{x}^T \mathbf{y} = 0$

**Facts** #1 $\cos\theta = \dfrac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$

#2 if $\mathbf{u}, \mathbf{v}$ are unit vectors $\cos\theta = \mathbf{u}^T \mathbf{v}$

#3 Cauchy - Schwarz Inequality

a) $-1 \le \dfrac{(\mathbf{x}^T \mathbf{y})}{\|\mathbf{x}\| \|\mathbf{y}\|} \le 1$

b) $\dfrac{|\mathbf{x}^T \mathbf{y}|}{\|\mathbf{x}\| \|\mathbf{y}\|} \le 1$

c) $|\mathbf{x}^T \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$

<u>and</u> equality only if a vector is $0$ or $\mathbf{x} = \alpha \mathbf{y}$
↑
<u>parallel</u>

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**$\mathbb{R}^n$** we can use $\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$
as a "non-physical" definition for $\mathbf{x} \perp \mathbf{y}$

$$\mathbf{x}^T \mathbf{y} = 0 \quad \text{iff} \quad \mathbf{x} \perp \mathbf{y}$$

<u>Properties</u> : <u>still have</u> $\cos\theta = \dfrac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$

b/c $|\mathbf{x}^T \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$

and law of cosines $\|y - x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$

if $x \perp y$ then $x^T y = 0$ then $\cos\theta = 0$

so for $\mathbb{R}^n$ $\|y - x\|^2 = \|x\|^2 + \|y\|^2$

Pythagorean Law