Math 511

Q's 4.2 example #4 $L([x]_B) = [y]_B$

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}_B\right) = \begin{bmatrix} a+b \\ 2b \end{bmatrix}_B$$

Matrix for L in basis B

$$\left[\, L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_B\right) \;\; L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_B\right) \right] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}_B \begin{bmatrix} 1 \\ 2 \end{bmatrix}_B \right] = A$$

4.2 figure 4.2.2 $[y]_D = D^{-1} A\, B\, [x]_B$

A is standard matrix for L

$x : [x]_B \longrightarrow B[x]_B = [x]_E \longrightarrow A(B[x]_B) = [y]_E \longrightarrow \boxed{D^{-1}}(A\,B[x]_B) = [y]_D : y$

go from basis B to standard coord

A is standard matrix of L (do lin. transform)

go from stand. to basis D

$$\begin{tikzcd}
E[v]_E = v \in V \arrow[r, "L = L_A"] \arrow[d, leftrightarrow] & w = L(v) \in W \arrow[d, leftrightarrow, "F^{-1}[w]"] \\
x = [v]_E \arrow[r, "A"] & Ax = [w]_F
\end{tikzcd}$$

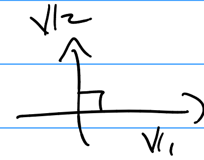
v

E ← not standard

F ← not standard

$F^{-1} L_A E$

$\boxed{\text{ch 5}}$ orthogonal (in $\mathbb{R}^n$)

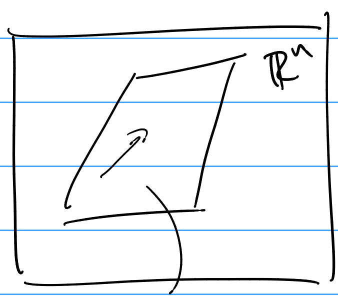
in $\mathbb{R}^2$, $\mathbb{R}^3$ has a physical concept 

$v_1 \perp v_2$

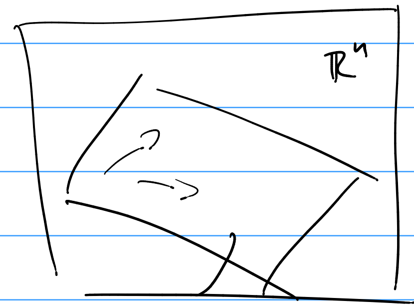
→ found a mathematical tool $v_1^T v_2 = 0$ then $v_1 \perp v_2$

$v_1^T v_2$: scalar product

$\boxed{5.2}$



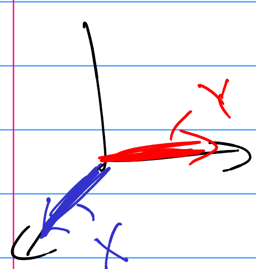
X is a subspace of $\mathbb{R}^n$



Y is a subspace of $\mathbb{R}^n$

Def: if every $x \in X$ and every $y \in Y$ have $x^T y = 0$
(so every $x \in X$ and y of Y are orthogonal)

ex



$X = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \text{span}(e_1)$

$Y = \text{span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \text{span}(e_2)$

any $x \in X$ is $\begin{bmatrix}\alpha\\0\\0\end{bmatrix}$
any $y \in Y$ is $\begin{bmatrix}0\\\beta\\0\end{bmatrix}$
$\Big\}\ x^T y = 0$

→ we call X, Y to be orthogonal subspaces

Def: if S is a subspace of $\mathbb{R}^n$. Collect all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in S..

$$S^\perp = \{ x \in \mathbb{R}^n \mid \text{for all } s \in S \quad x^T s = 0 \}$$

is the orthogonal complement of S.

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ex5 $\mathbb{R}^5$

$$X = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \quad \text{any } x = \begin{bmatrix} 0 \\ \alpha \\ \beta \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{any } y = \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ c_2 \end{bmatrix}$$

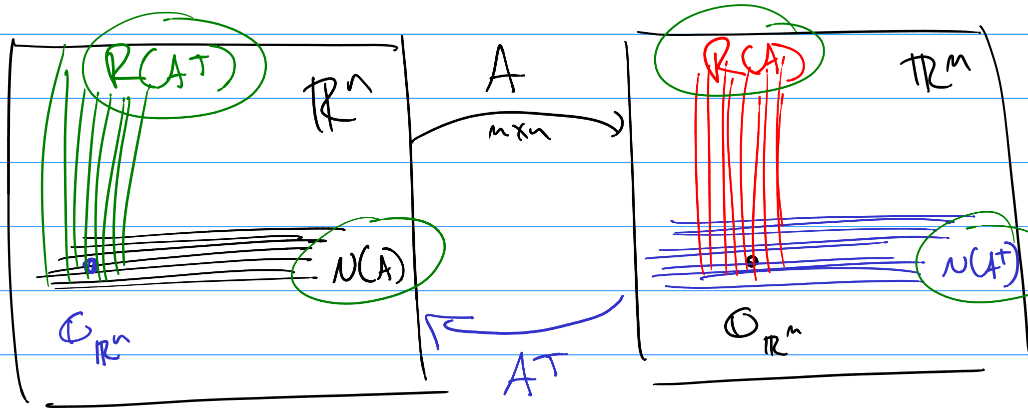
$x^T y = 0$

So X, Y are orthogonal.

but $$X^\perp = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$\boxed{\text{Fundamental Subspaces}}$ → Fundamental orthogonal complements subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ created by A $(m \times n)$ matrix.

$R(A^T)$ is range of A^T
=
Col. space of A^T
=
row space of A written as column vectors



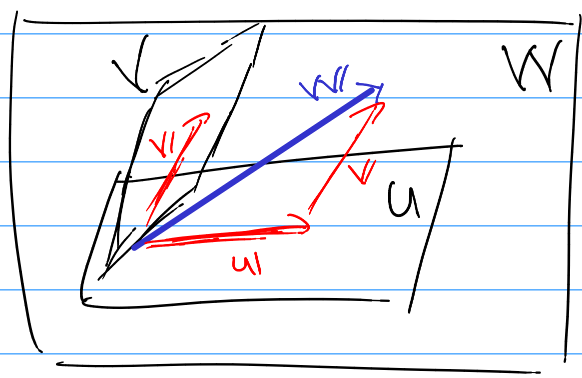
$R(A)$ is the range of A
=
Col. space of A

Thm $N(A) = R(A^T)^\perp$, $N(A^T) = R(A)^\perp$

Def: if U, V are subspaces of W and every $w \in W$ can be uniquely written as $u + v = w$, $u \in U$, $v \in V$

then W is called a direct sum of U, V

$$W = U \oplus V$$



Thm for any subspace S of $\mathbb{R}^n$

$$\mathbb{R}^n = S \oplus S^\perp$$

Thm $(S^\perp)^\perp = S$

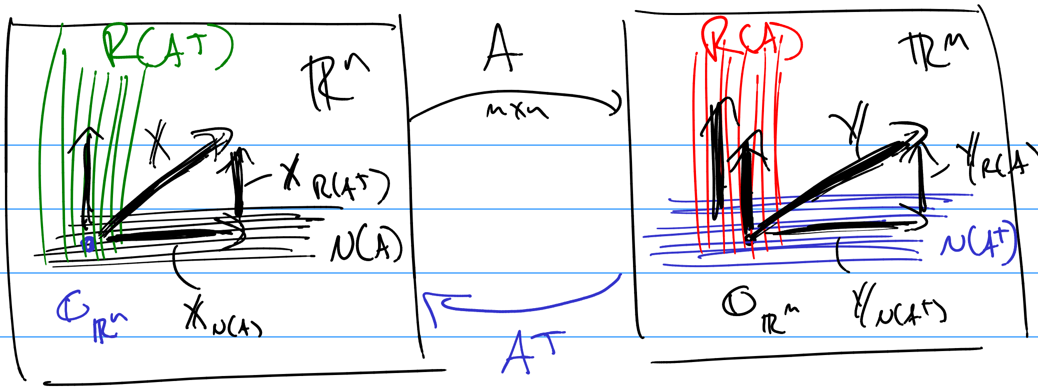
So $N(A) = R(A^T)^\perp$ and $N(A)^\perp = R(A^T)$; $\mathbb{R}^n$

$N(A^T) = R(A)^\perp$ and $N(A^T)^\perp = R(A)$: $\mathbb{R}^m$

therefore $\mathbb{R}^n = N(A) \oplus R(A^T)$

$\mathbb{R}^m = N(A^T) \oplus R(A)$

$R(A^T)$ is range of A^T
$=$
Col. space of A^T
$=$
row space of A written as column vectors

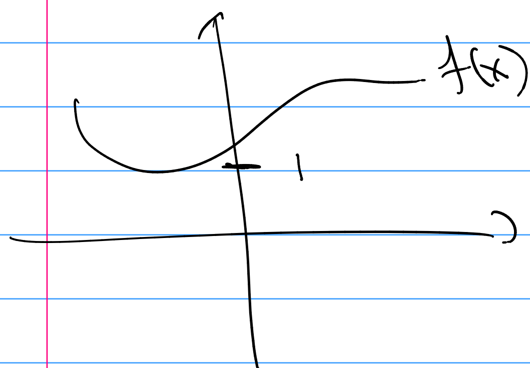


$R(A)$ is the range of A
$=$
Col. space of A

$n \times m$

$$\mathbb{R}^n = N(A) \oplus R(A^T) \qquad \mathbb{R}^m = N(A^T) \oplus R(A)$$

$$x = x_{N(A)} + x_{R(A^T)} \qquad y = y_{N(A^T)} + y_{R(A)}$$



find x such that $f(x) = -1$

$\mathbb{R}^3$ in $\mathbb{R}^3$ $\qquad (\mathbb{R}^3)^\perp = \{0\}$

$\dim(S) = 2$

$$\underset{n}{S} \oplus \underset{0}{S^\perp} = \mathbb{R}^n$$