Math 511

$\boxed{\text{CQ's}}$ 5.2 #3

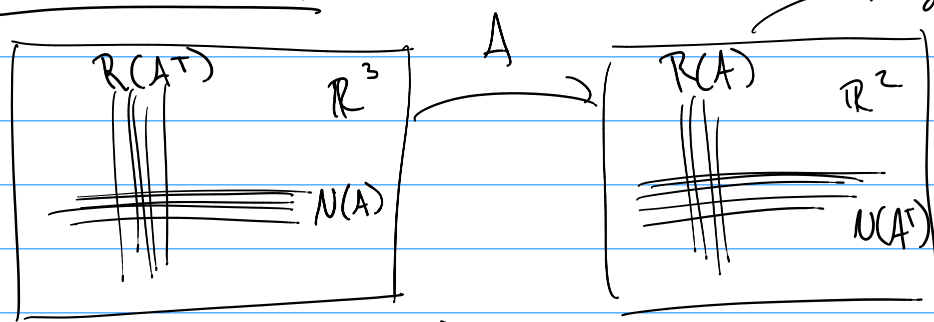
$S = \text{Span}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) =$ Col. space of A^T = row space of A as col. = $R(A^T)$

$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}_{2\times 3}$ Show $N(A) = S^\perp$

$\boxed{\text{Know}}$ Fundamental Subspaces:

Range of A^T = row space of A as col's = col space of A^T

range of A = col. space of A



$\mathbb{R}^3 = N(A) \oplus R(A^T)$ $\qquad$ $\mathbb{R}^2 = N(A^T) \oplus R(A)$

$N(A) = R(A^T)^\perp$

$N(A)^\perp = R(A^T)$

by observation we see $S = \text{Span}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$

with $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ see S is A's row space as col's = $\boxed{R(A^T) = S}$

a) b/c $R(A^T)^\perp = N(A)$, $R(A^T) = S$

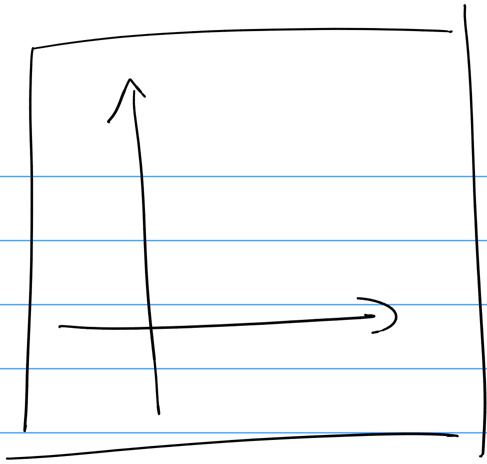
$S^\perp = N(A)$

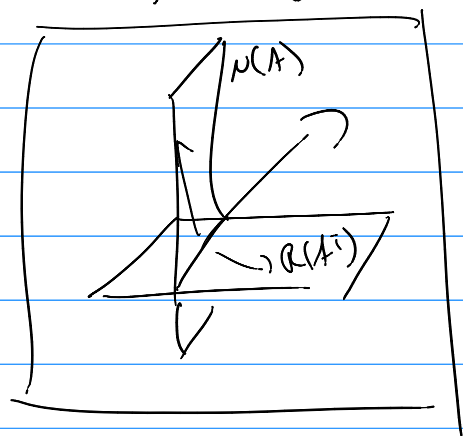
b) $S = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right)$ $\qquad$ $S^\perp = ?? = N(A)$, $S = R(A^T)$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

$N(A)?$: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \rightarrow$ Solve!

ways to look at fund. subspaces





p. 145 D is derivative (linear transform) $P_3 \to p = a + bx + cx^2$

$p = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

linear transform.

Standard basis: $[1, x, x^2]$

non standard $[1, 2x, 4x^2-2] \to B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Standard matrix: $[\,L(P_1) \;\; L(P_2) \;\; L(P_3)\,] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = A$

$\frac{d}{dx}[1] = 0 = 0 + 0x + 0x^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{d}{dx}[x] = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\frac{d}{dx}[x^2] = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

Know: basis B on linear operator

$$[y]_B = B^{-1} A B [x]_B$$

$$\left[\frac{d}{dx}[\mathbb{P}]\right]_B = \underbrace{\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}^{-1}\underbrace{\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}}_{\text{derivative if } \mathbb{P} \text{ is in standard}}\begin{bmatrix}1&0&-2\\0&2&0\\0&0&4\end{bmatrix}}\;[\mathbb{P}]_B$$

ex $\quad \frac{d}{dx}\{2+3x-2x^2\} = 3-4x$

$$\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}2\\3\\-2\end{bmatrix} = \begin{bmatrix}3\\-4\\0\end{bmatrix}$$

ex $\quad \frac{d}{dx}\underbrace{\{3(1)-2(2x)+3(4x^2-2)\}}_{\begin{bmatrix}3\\-2\\3\end{bmatrix}_B}$

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ex $\quad L: \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad$ Linear operator $\qquad [y]_B = \boxed{B^{-1}AB}[x]_B$

$$B = \begin{bmatrix}1&-2&1\\-1&1&1\\0&1&1\end{bmatrix} \qquad AB = \begin{bmatrix}3&3&0\\1&1&2\\1&1&2\end{bmatrix}\begin{bmatrix}1&-2&1\\-1&1&1\\0&1&1\end{bmatrix}$$

$$[L(b_1)\ L(b_2)\ L(b_3)] = AB = \begin{bmatrix}0&-2&4\\0&1&4\\0&1&4\end{bmatrix}$$

$$B^{-1}AB \to \left[\begin{array}{ccc|ccc}1&-2&1&0&-2&4\\-1&1&1&0&1&4\\0&1&1&0&1&4\end{array}\right]$$

$$\left[\, I \mid B^{-1}AB \,\right]$$

check: $\left[\, I \;\middle|\; \begin{matrix}0&0&0\\0&1&0\\0&0&4\end{matrix}\right]$

$$L(v_1) = Av = 0v_1 + 0v_2 + 0v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_B$$

$$L(v_2) = 0v_1 + 1v_2 + 0v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B$$

$$L(v_3) = \underline{0v_1 + 0v_2 + 4v_3} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}_B$$

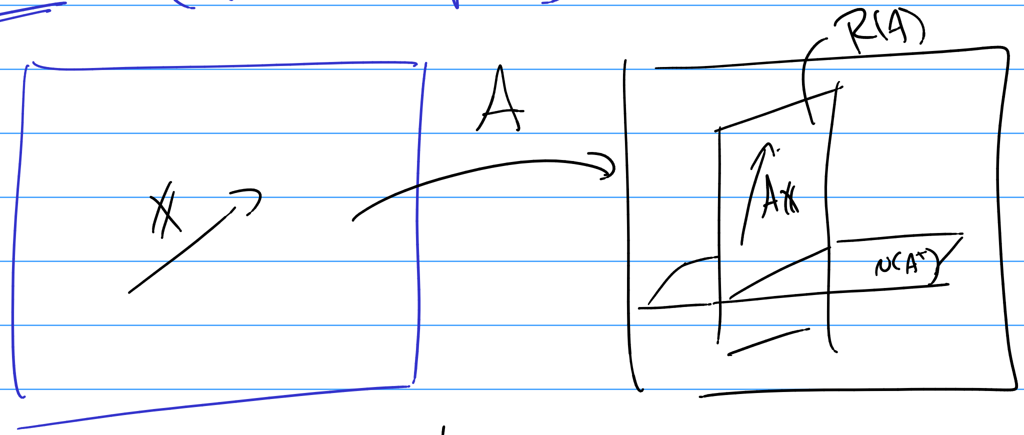
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = B^{-1} A B$$

$$B \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} B^{-1} = A$$

standard to standard L

$$\underline{\begin{bmatrix} ? & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}$$

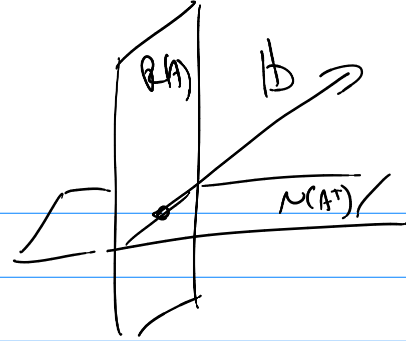
Application: (Fund. Subspaces)



Solve: $Ax = b$

Know: consistent if and only if $b \in R(A)$

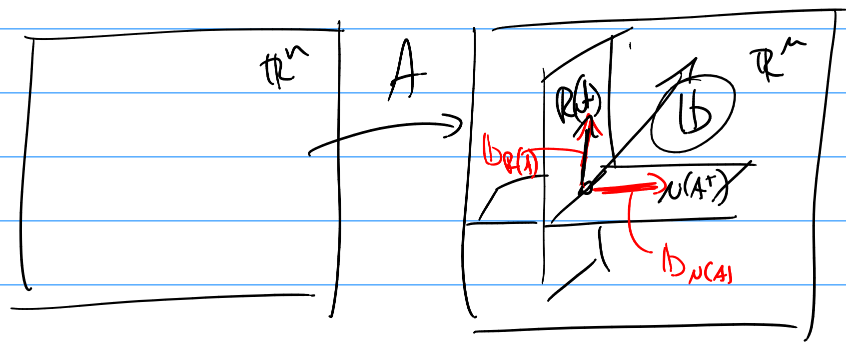
but what if



$b \notin R(A)$

before we just said no solution

Now: use



by fund. subspaces $b = b_{N(A^T)} + b_{R(A)}$

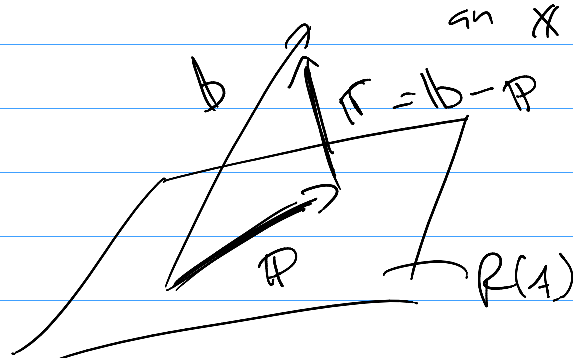
what happens if $A^T b = \underbrace{A^T (b_{N(A^T)}}_{0} + b_{R(A)})$

so $A^T b = A^T b_{R(A)}$

this does have an x so $Ax =$ it

least squares solution:

$Ax = b$ has no soln.



goal minimize $\|r\| = \|b - p\|$

Same as: minimize $\|r\|^2$

least squares solution.

How to find least squares solution?

$$\underset{m\times n}{A}\,x = b \quad \text{has no solution}$$

so **Solve**

$$\boxed{\underset{n\times n}{A^T A}\,x = A^T b}$$

$$\hat{x} = \boxed{(A^T A)^{-1}(A^T b)}$$

then $A\hat{x} = p = \underline{(A(A^TA)^{-1}A^T)}\, b$

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example:

$3x + y + z = 1$
$x - y + 2z = 4$
$\quad 2y - z = -1$
$4x \quad + z = 3$
$x + y + z = 1$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & -1 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Check no sol.

no sol but find x, y, z to get "close" — least squares

Solve this!

$$\underset{A^T}{\begin{bmatrix} 3 & 1 & 0 & 4 & 1 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix}}\underset{A}{\begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & -1 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}\underset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{A^T}{\begin{bmatrix} 3 & 1 & 0 & 4 & 1 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 1 & 1 \end{bmatrix}}\underset{b}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \\ 1 \end{bmatrix}}$$

$$\underset{3\times 3}{\overset{(A^TA)}{\begin{bmatrix} 27 & 4 & 10 \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}}}\underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3\times 1}{\overset{(A^Tb)}{\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}}} \text{ find}$$

$\rightarrow$ better!

#7 of 4.2

4.1 $\qquad \Upsilon = \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}$

any vector space $I(v) = v$ is the identity operator.

$$I(e_1) = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}_{E} \overset{?}{=} \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_{\Upsilon}$$

standard

change of coord. from standard to Υ

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_{\Upsilon} = \Upsilon^{-1}[x]_E$$

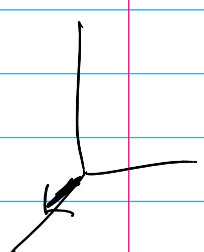
a) find coord. of $I(e_1), I(e_2), I(e_3)$ w.r.t. Υ

i) $I(e_1)$ in Υ coord.

$I(e_1) = e_1$ by def. of I

$$e_1 = \begin{bmatrix}?\end{bmatrix}_{\Upsilon} \quad \underline{\text{use}} \quad \Upsilon^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}?\end{bmatrix}_{\Upsilon}$$

$$\Upsilon = \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix} \qquad \Upsilon^{-1} = \begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}$$



$$\Upsilon^{-1}e_1 = \begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}_{\Upsilon}$$

$$\Upsilon^{-1}e_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}_{\Upsilon}$$

$$\Upsilon^{-1}e_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}_{\Upsilon}$$

b) $$\begin{bmatrix}\Upsilon^{-1}e_1 & \Upsilon^{-1}e_2 & \Upsilon^{-1}e_3\end{bmatrix} = \Upsilon^{-1} = \underline{\underline{\begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}}}$$

4.3 #6

$V = \text{span}(1, e^x, e^{-x}) = c_1(1) + c_2(e^x) + c_3(e^{-x})$

$= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\text{standard}}$

Standard Basis

$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e^x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$e^{-x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Non-Standard $[1, \cosh x, \sinh x] = \left[1, \frac{e^x + e^{-x}}{2}, \frac{e^x - e^{-x}}{2}\right]$

$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\cosh x = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$

$\sinh x = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$

linear operator: Derivative:

deriv(1) deriv(e^x) deriv(e^{-x})

Standard Matrix $[\, L(P_1) \quad L(P_2) \quad L(P_3) \,]$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(continue)