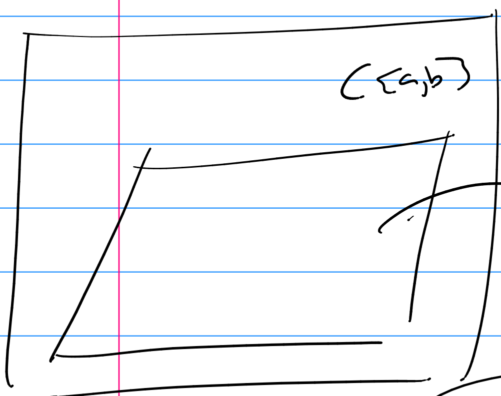
Math 511

Q's 4.3 #6

borrow P_n idea

ex $p = 1 + 2x + x^2 = \begin{array}{c} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow 1 \\ \leftarrow x \\ \leftarrow x^2 \end{array}$



our vectorspace

$$S = \text{span}(1, e^x, e^{-x})$$

so $3 - \pi e^x + \sqrt{2}\, e^{-x} = \begin{bmatrix} 3 \\ -\pi \\ \sqrt{2} \end{bmatrix}$

Stand basis: $1, e^x, e^{-x} \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

non-standard: $1, \cosh x, \sinh x \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$

$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

Linear Operator is Derivative:

$\frac{d}{dx}(1) = 0$, $\frac{d}{dx}[e^x] = e^x$, $\frac{d}{dx}[e^{-x}] = -e^x$

$\to$ standard matrix $\begin{bmatrix} L(1), & L(e^x), & L(e^{-x}) \end{bmatrix}$

as vectors $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

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inner product space: V a vector space with $\langle u, v \rangle$ defined

then: ① norm / magnitude / "length" $\|v\|^2 = \langle v, v \rangle$

or

$$\|v\| = (\langle v, v \rangle)^{1/2}$$

② $u \perp v$ if and only if $\langle u, v \rangle = 0$

③ Cauchy – Schwarz $\quad |\langle u, v \rangle| \le \|u\| \, \|v\|$

or

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|}$$

ex $\quad C[-1,1] \qquad \langle f, g \rangle = \int_{-1}^{1} fg \, dx$

θ between $\sin x$ and $1+x^2$?

Step 1 $\quad \langle \sin x, 1+x^2 \rangle = \int_{-1}^{1} (\sin x)(1+x^2) \, dx = 0$

Step 2 $\quad \|\sin x\| = \left(\langle \sin x, \sin x \rangle\right)^{1/2} = \left(\int_{-1}^{1} \sin^2 x \, dx \right)^{1/2} =$

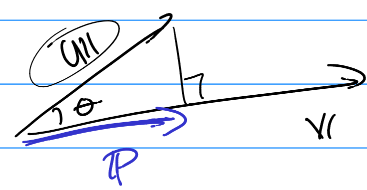
Step 3 $\quad \|1+x^2\| = \left(\langle 1+x^2, 1+x^2 \rangle\right)^{1/2} = \left(\int_{-1}^{1} (1+x^2)^2 \, dx \right)^{1/2} =$

Projections $\quad P$ is the projection of u onto v

use: 

$\cos\theta = \frac{a}{c}$

$a = c \cos\theta$

So → 

$\|p\| = \|u\| \cos\theta$

but $\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|}$

Step 1 $\quad \alpha = \|p\| = \boxed{\frac{\langle u, v \rangle}{\|v\|}}$ $\quad$ scalar projection

Step 2 $\quad P = \alpha \frac{1}{\|v\|} v = \left(\frac{\langle u, v \rangle}{\langle v, v \rangle} \right) v$

(9c) Project $\overset{U}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}}$ onto $\overset{V}{\begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}}$

use $\mathbb{R}^{2\times 3}$ vector space w/ $\langle A, B\rangle = \sum\sum a_{ij} b_{ij}$

Step 1 $\alpha = \|P\| = \dfrac{\left\langle \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix} \right\|} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

Step 2 $P = \alpha \dfrac{1}{\|V\|} V = \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{3\sqrt{2}}\right)\begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ 1/6 & 0 & 1/2 \end{bmatrix}$

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(1) Inner Product Space gives "angles" and "magnitudes"

(2) If you only are interested in "magnitudes" we can make a $\boxed{\text{Normed linear Space}}$

take V and add a unary operator $\|V\| = \text{Scalar}$

where (1) $\|V\| \geq 0$ and only $\|0\| = 0$

(2) $\|\alpha V\| = |\alpha|\,\|V\|$

(3) $\|V_1 + V_2\| \leq \|V_1\| + \|V_2\|$

$\mathbb{R}^n$ typical norm used is p-norm

$$\|X\|_p = \left(|X_1|^p + |X_2|^p + \dots + |X_n|^p\right)^{1/p}$$

So our older norm ($\sqrt{\text{sum squares}}$) $\|X\|_2 = \sqrt{\langle X, X\rangle}$

as $p \to \infty$ $\|X\|_\infty = \max(|X_1|, |X_2|, \dots, |X_n|)$

$\boxed{5.5}$ V with $\langle u, v \rangle$ defined

Orthonormal Set $v_1, v_2, \dots, v_k$ if

#1 $\langle v_i, v_j \rangle = 0$ when $i \neq j$

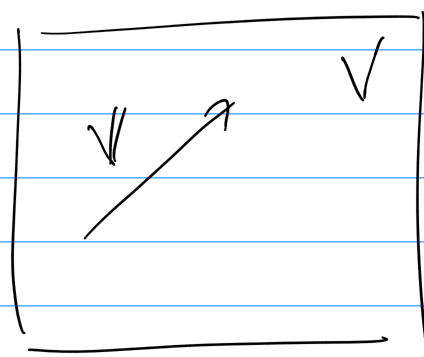
so $v_i \perp v_j$ when $i \neq j$] $\{v_1, \dots, v_k\}$ **orthogonal set**

#2 $\| v_i \| = 1$

call $\{v_1, \dots, v_k\}$ **orthonormal set.**

Def: if a basis is orthonormal as well .. call it a orthonormal basis for V.

Why are these sets interesting?



$\dim(V) = n$

$v = [\text{coord in basis}]_B$

$B = [b_1\ b_2 \cdots b_n]$

$\boxed{\text{Thm}}$ if $U = [u_1\ u_2 \cdots u_n]$ is an orthonormal basis then

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_U$$

coord. in basis U

$c_1 = \langle v, u_1 \rangle$

$c_2 = \langle v, u_2 \rangle$

$\vdots$

$c_n = \langle v, u_n \rangle$

so $v = \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \vdots \\ \langle v, u_n \rangle \end{bmatrix}_U$