Math 511

L a linear operator $\quad L: V \to V$

"with respect to?"

$$X_B \;(V) \xrightarrow{\text{Matrix?}} Y_E \;(V)$$

- Standard
- Basis B

- Standard
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L as a standard matrix $= [L_{stand}]$

operator: $L([X]_E) = [Y]_E$

Matrix mult.: $[L_{stand}][X]_E = [Y]_E$

① Matrix. Operator from stand to stand $\quad [L_{stand}]$

② Matrix of operator from B to B $\quad B^{-1}[L_{stand}]B = [L_{non\ stand}]$

③ matrix of operator from B to stand. $\quad [L_{stand}]B = B[L_{non-stand}]$

④ Matrix of operator from stand to B $\quad B^{-1}[L_{stand}] = [L_{non\text{-}stand}]B^{-1}$

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ex $\quad L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}_B\right) = \begin{bmatrix} a+b \\ b+c \\ -c \end{bmatrix}_B \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$[L_{non\text{-}stand}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[Y]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}[X]_B$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$[L_{stand}] = B \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} B^{-1}$$

$$[L_{stand}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

L from stand to stand

② L from stand to B?

$$[y]_B = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} B^{-1}}_{L_{non\text{-}stand}} [x]_E$$

$$\boxed{[L \text{ from stand to } B] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}}$$

orthonormal sets $\quad v_1, v_2, \ldots, v_k$

① orthogonal set if $\langle v_i, v_j \rangle = 0$ for $i \neq j$

② orthonormal set if a) $\langle v_i, v_j \rangle = 0$ for $i \neq j$

b) $\| v_i \| = 1$

③ basis is also orthonormal. call it an **orthonormal basis**

$C[-\pi,\pi]$ $\quad \langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

ex $\quad f_1 = \frac{1}{\sqrt{2}},\ f_2 = \sin x,\ f_3 = \cos x,\ \dots,\ f_{2n+1} = \cos nx$

check: show #1 f_i has $\|f_i\| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}(f_i)^2\,dx}$

$$\|f_i\| \overset{?}{=} 1$$

#2 $\quad \langle f_i, f_j\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f_i f_j\,dx \overset{?}{=} 0$

so (if you do these integrals) we know

$X = c_1\frac{1}{\sqrt{2}} + c_2\sin x + c_3\cos x \qquad [X]_u$

$\longrightarrow \frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin nx, \cos nx$

are an orthonormal set.

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why? (properties $\to$ nice applications)

#1 $\boxed{\text{thm}}$ an orthonormal set is also linearly independent.

#2 given an orthonormal basis $U = [u_1\ u_2 \dots u_n]$

so $\quad X = c_1u_1 + c_2u_2 + \dots + c_nu_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n\end{bmatrix}_u$

each $\quad c_i = \langle X, u_i\rangle$

Corollary $\qquad X = [X]_u \qquad Y = [Y]_u$

then ① $\langle X, Y\rangle = [X]_u^T\,[Y]_u = x_1y_1 + x_2y_2 + \dots + x_ny_n$

$\langle f, g \rangle =$

ex $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\overbrace{\underbrace{3\left(\tfrac{1}{\sqrt{2}}\right) + (-2)\sin x + (3)\cos x}_{\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}_u}}^{f}\Big)\Big(\overbrace{\underbrace{(-1)\left(\tfrac{1}{\sqrt{2}}\right) + (-3)\sin x + (-1)\cos x}_{\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}_u}}^{g}\Big)\,dx$

$$= (3)(-1) + (-2)(-3) + (3)(-1) = 0$$

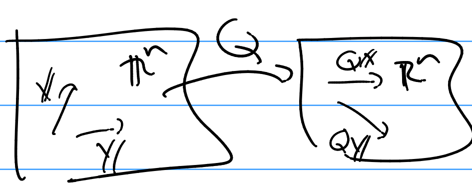
\# $\|x\|^2 = \langle x, x \rangle = [x]_u^T [x]_u$

ex $\Big\| \underbrace{3\left(\tfrac{1}{\sqrt{2}}\right) + (-2)\sin(x) + 3\cos(x)}_{\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}_u} \Big\|^2 = 9 + 4 + 9 = \boxed{22}$

Orthogonal Matrix $\quad Q = [q_1, q_2 \cdots q_n]$ $\quad q_i$ are orthonormal

Properties:

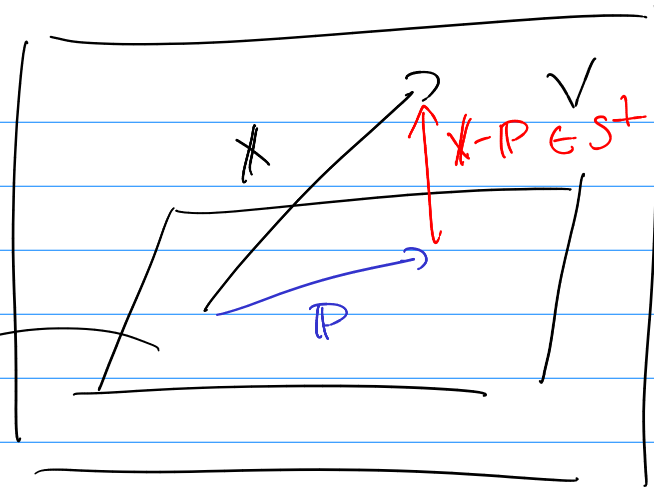
1. $q_1, q_2, \ldots, q_n$ are an orthonormal basis for $\mathbb{R}^n$
2. $Q^T Q = I$
3. $Q^{-1} = Q^T$



4. $\langle Qx, Qy \rangle = \langle x, y \rangle$
5. $\|Qx\|_2 = \|x\|_2$

ex E type 1 ex $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

read p. 259 – 263



$S = \mathrm{span}(\underbrace{u_1, u_2, \ldots u_k}_{\text{orthonormal basis}})$

$$p = [p]_u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}_u$$

where $c_i = \langle x, u_i \rangle$

① $p \in S$

② $x - p \in S^\perp$

③ p is closest to x in S

p: projection of x onto S.

Fourier Transform

projection of $f(x)$ onto $\mathrm{span}\left(\frac{1}{\sqrt{2}}, \sin x, \cos x, \ldots, \sin(nx), \cos(nx)\right)$