Math 511

Q's

$X_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $X_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

a) orthonormal?

① show $X_1 \perp X_2$ by $\langle X_1, X_2 \rangle = 0$

$(\cos(\theta))(-\sin\theta) + (\sin\theta)(\cos\theta) \overset{?}{=} 0$

② $\|X_1\| \overset{?}{=} 1$, $\|X_2\| \overset{?}{=} 1$ } show!

a) $\dim(\mathbb{R}^2) = 2$

X_1, X_2 are orthogonal $\rightarrow$ X_1, X_2 are lin. ind.

so basis!

b) $Y = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_X = \begin{bmatrix} \langle Y, X_1 \rangle \\ \langle Y, X_2 \rangle \end{bmatrix}_X = \langle Y, X_1 \rangle X_1 + \langle Y, X_2 \rangle X_2$

Note: if $u_1, u_2, \ldots, u_k$ are orthonormal basis

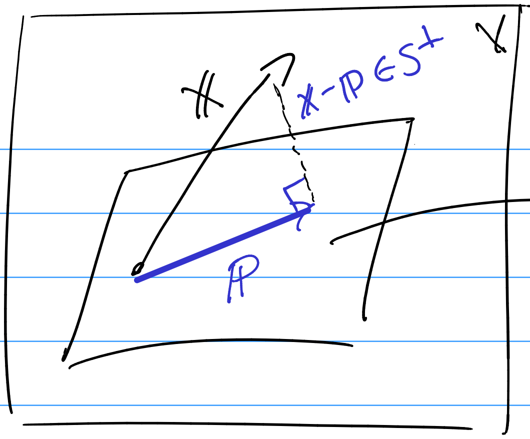
$$Y = \begin{bmatrix} \langle Y, u_1 \rangle \\ \langle Y, u_2 \rangle \\ \vdots \\ \langle Y, u_k \rangle \end{bmatrix}_u = \langle Y, u_1 \rangle u_1 + \langle Y, u_2 \rangle u_2 + \cdots + \langle Y, u_k \rangle u_k$$

So $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $X_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $X_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

so $Y = \langle Y, X_1 \rangle X_1 + \boxed{\langle Y, X_2 \rangle} X_2$

$= \underbrace{(y_1\cos\theta + y_2\sin\theta)}_{c_1} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \underbrace{(-y_1\sin\theta + y_2\cos\theta)}_{c_2} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

For



$$\underline{S} = \text{Span}(\overbrace{u_1, u_2, \ldots, u_k}^{\text{orthonormal}})$$

① $\mathbb{P} = \begin{bmatrix} \langle \mathbb{X}, u_1 \rangle \\ \langle \mathbb{X}, u_2 \rangle \\ \vdots \\ \langle \mathbb{X}, u_k \rangle \end{bmatrix}_{U} = \underbrace{\langle \mathbb{X}, u_1 \rangle}_{\substack{\mathbb{X}^T u_1 \\ u_1^T \mathbb{X}}} u_1 + \underbrace{\langle \mathbb{X}, u_2 \rangle}_{\substack{\mathbb{X}^T u_2 \\ u_2^T \mathbb{X}}} u_2 + \cdots + \langle \mathbb{X}, u_k \rangle u_k$

then $\mathbb{P} \in S$

out ② $\mathbb{X} - \mathbb{P} \in S^{\perp}$

③ $\mathbb{P}$ is element in S closest to $\mathbb{X}$.

Note: If we are studying $\mathbb{R}^n$

$$\mathbb{P} = \begin{bmatrix} \langle \mathbb{X}, u_1 \rangle \\ \langle \mathbb{X}, u_2 \rangle \\ \vdots \\ \langle \mathbb{X}, u_k \rangle \end{bmatrix}_{U} = \begin{bmatrix} U^T \mathbb{X} \end{bmatrix}_{U}$$

$$\mathbb{P} = U U^T \mathbb{X}$$

matrix that projects $\mathbb{X}$ onto span$(u_1, u_2, \ldots u_k)$

b/c orthonormal basis are $\boxed{\text{very}}$ useful .. how to make them?

5.6 Gram-Schmidt Orthogonalization

start with non-orthogonal non-length one vectors (need simply linear independ)

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \xrightarrow[\text{Schmidt}]{\text{gram}} \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n \leftarrow \text{orthonormal}$$

and $[\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_n] = [\mathbf{q}_1\ \mathbf{q}_2 \ldots \mathbf{q}_n] R$

upper-triangular

$$X = QR$$

why? #1 orthonormal vectors have applications for projecting into nice spans (ex Fourier or orthog. polynomials)

#2 Solve $A\mathbf{x} = \mathbf{b}$

$A = QR$

orthogonal so $Q^{-1} = Q^T$

$$QR\mathbf{x} = \mathbf{b}$$

$$R\mathbf{x} = Q^T\mathbf{b}$$

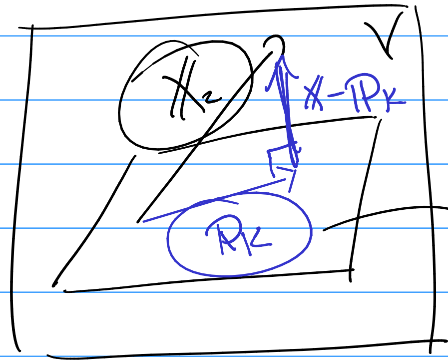
upper triangular so we can solve this with simple back solve.

$$\boxed{\hat{\mathbf{x}} = \text{least square soln.}}$$

Gram-Schmidt

recursive process.

concept



$\longrightarrow \text{Span}(q_1, q_2, \ldots, q_K)$

$$P_K = \begin{bmatrix} \langle X, q_1 \rangle \\ \langle X, q_2 \rangle \\ \vdots \\ \langle X, q_K \rangle \end{bmatrix} Q = \langle X, q_1 \rangle q_1 + \langle X, q_2 \rangle q_2 + \cdots + \langle X, q_K \rangle q_K$$

want? not in span!. take $X - P_K$

$$\text{next } q_{K+1} = \frac{1}{\|X - P_K\|}(X - P_K)$$

orthogonal and length 1.

$$X_1, X_2, \ldots, X_n \longrightarrow q_1, q_2, \ldots, q_n$$

Base Step: $q_1 = \frac{1}{\|X_1\|} X_1$

Inductive Step is a loop.. $X_2 \to q_2$, $X_3 \to q_3$, .., $X_{K+1} \to q_{K+1}$

$X_2 \to q_2$

a) $P_1 = \langle X_2, q_1 \rangle q_1$

b) $q_2 = \frac{1}{\|X_2 - P_1\|}(X_2 - P_1)$

know q_1, q_2

$X_3 \to q_3$

a) $P_2 = \langle X_3, q_1 \rangle q_1 + \langle X_3, q_2 \rangle q_2$

b) $q_3 = \frac{1}{\|X_3 - P_2\|}(X_3 - P_2)$

know q_1, q_2, q_3

$x_4 \to q_4$ a) $p_3 = \langle x_4, q_1 \rangle q_1 + \langle x_4, q_2 \rangle q_2 + \langle x_4, q_3 \rangle q_3$

b) $q_4 = \frac{1}{\|x_4 - p_3\|}(x_4 - p_3)$ Know: q_1, q_2, q_3, q_4

$\vdots$

Note: for $\mathbb{R}^n$

$$[x_1 \; x_2 \; \cdots \; x_n] = [q_1 \; q_2 \; \cdots \; q_n] \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \left[\begin{array}{c} \|x_1\| \\ 0 \\ \vdots \\ 0 \end{array}\right. & \begin{array}{c} \langle x_2, q_1 \rangle \\ \|x_2 - p_1\| \\ 0 \\ \vdots \\ 0 \end{array} & \begin{array}{c} \langle x_3, q_1 \rangle \\ \langle x_3, q_2 \rangle \\ \|x_3 - p_2\| \\ 0 \\ \vdots \end{array} & \left.\begin{array}{c} \cdots \\ \\ \\ \\ \\ \end{array}\right] \end{array}$$

$$X = QR$$

Numerical Methods

study of operations and how error can be controlled or not controlled.

ex

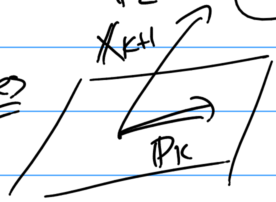
gram-schmidt vs modified gram-schmidt

$x_1 \to q_1 \to \|x_1\|$ $\cancel{x_1}$

x_2 p_1 q_2

x_3 p_2 q_3

uses



$x_2 \longrightarrow q_2$ also $x_3 \to$ New x_3

$x_4 \to$ New x_4

$\vdots$

$x_n \to$ new x_n