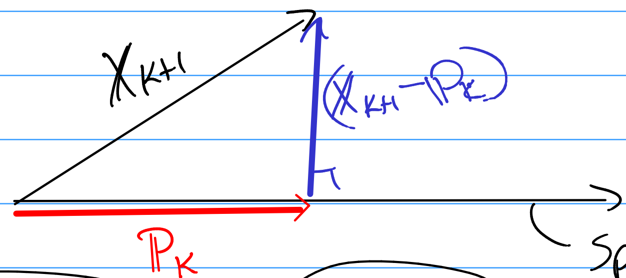
Math 511

Q's 5.6 #5

Gram-Schmidt (concept)

$X_{K+1} \to q_{K+1}$



know q_1 to q_K orthonormal vectors

$\underbrace{\mathrm{Span}(q_1, q_2, \dots, q_K)}$

Know

a) $P_K = \langle X_{K+1}, q_1 \rangle q_1 + \langle X_{K+1}, q_2 \rangle q_2 + \cdots + \langle X_{K+1}, q_K \rangle q_K$

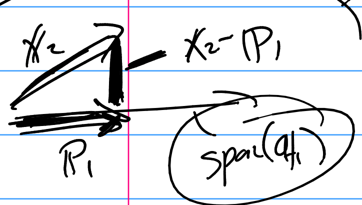
b) $X_{K+1} - P_K \to q_{K+1} = \dfrac{1}{\|X_{K+1} - P_K\|}(X_{K+1} - P_K)$

#1 $X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \to q_1 = ?$ $q_2 = ?$ $R = \begin{bmatrix} ? \end{bmatrix}$

$X_1 \to q_1$ $q_1 = \dfrac{1}{\|X_1\|} X_1 = \dfrac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

#2

$X_2 \to q_2$



a) $P_1 = \langle X_2, q_1 \rangle q_1 = \dfrac{5}{3}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 5/9 \\ 10/9 \end{bmatrix}$

b) $X_2 - P_1 = \begin{bmatrix} -1/9 \\ 4/9 \\ -1/9 \end{bmatrix}$

$q_2 = \dfrac{1}{\frac{\sqrt{18}}{9}}\begin{bmatrix} -1/9 \\ 4/9 \\ -1/9 \end{bmatrix} = \dfrac{3}{\sqrt{2}}\begin{bmatrix} -1/9 \\ 4/9 \\ -1/9 \end{bmatrix} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1/3 \\ 4/3 \\ -1/3 \end{bmatrix}$

$\dfrac{1}{\sqrt{2}/3}$

$q_2 = \begin{bmatrix} -1/3\sqrt{2} \\ 4/3\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}$

$$x_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow q_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \quad q_2 = \begin{bmatrix} -1/3\sqrt{2} \\ 4/3\sqrt{2} \\ -1/3\sqrt{2} \end{bmatrix}$$

$$X = QR$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3\sqrt{2} \\ 1/3 & 4/3\sqrt{2} \\ 2/3 & -1/3\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}$$

Solve $\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{b}$ $\qquad \underset{m\times n}{A} = \underset{m\times n}{Q}\ \underset{n\times n}{R}$

$$QRx = b$$

Q is orthogonal

$$\underset{n\times m}{Q^T}\ \underset{m\times n}{Q} = \underset{n\times n}{I}$$

$$\times \quad Q^T Q R x = Q^T b$$

$$\underset{n\times n}{R}\ \underset{n\times 1}{x} = \underbrace{(\underset{n\times m}{Q^T}\ \underset{m\times 1}{b})}_{n\times 1} \rightarrow \underline{\text{back solve!}}$$

$\hat{x} \leftarrow$ least squares soln.

Ex $\quad$ Try $\quad x_1 = 1 \quad x_2 = x \quad x_3 = x^2 \qquad \langle v_1, v_2\rangle = \int_{-1}^{1} v_1 v_2\, dx$

$$q_1 = ?_0 \quad q_2 = ?_0 \quad q_3 = ?_0$$

$x_1 \rightarrow q_1 \qquad q_1 = \frac{1}{\|x_1\|} x_1 \qquad \|1\| = \sqrt{\int_{-1}^{1} 1^2\, dx}$

$$q_1 = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \qquad = \sqrt{2}$$

$x_2 \rightarrow q_2 \qquad$ a) $\quad p_1 = \langle x_2, q_1\rangle q_1$

$$\int_{-1}^{1} (x)\left(\tfrac{1}{\sqrt{2}}\right) dx = 0$$

b) $\frac{1}{c}x$ is orthogonal to $\frac{1}{\sqrt{2}}$

$$\|x\| = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{2/3}$$

$$q_2 = \frac{1}{\|x\|} x$$

$$q_2 = \sqrt{3/2}\, x$$

$x_3 \to q_3$

a) $P_2 = \langle x_3, q_1\rangle q_1 + \langle x_3, q_2\rangle q_2$

b) $x_3 - P_2 \to q_3 = \frac{1}{\|x_3 - P_2\|}(x_3 - P_2)$

Exam 3

10 probs @ 10pts

10 pts = 100%

ch4 / ch5

4.1 (1 prob)

Is $L(\ ) = (\ )$ a linear transform / operator $L: V \to W$?

Matrix Space or $\mathbb{R}^n$

check: $L(\alpha_1 v_1 + \alpha_2 v_2) \overset{?}{=} \alpha_1 L(v_1) + \alpha_2 L(v_2)$

$$L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ d+e & e+f \end{bmatrix}$$

$V = \mathbb{R}^{2\times 3} \to W = \mathbb{R}^{2\times 2}$

4.2 Matrices representing Linear transforms. P_n

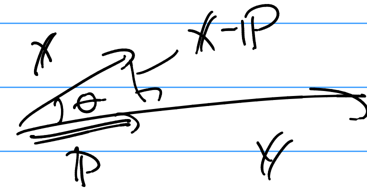
(1 prob) parts $\to$ given $L: V \to W$

a) L as a matrix from stand. basis to stand. basis

b) L as a matrix from non-stand. basis to non-stand. basis.

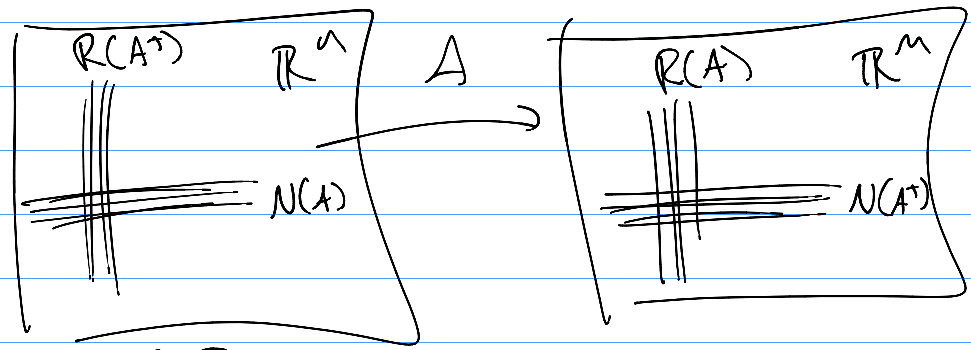
5.1 orthogonal for $\mathbb{R}^n$ $\quad x^T y = 0$

1 prob $\quad$ some $x^T y$ application $\rightarrow$



5.2 given A an $m \times n$ matrix $\quad A \rightarrow U$

draw:



$\rightarrow R(A^T) = \text{span}(?)$

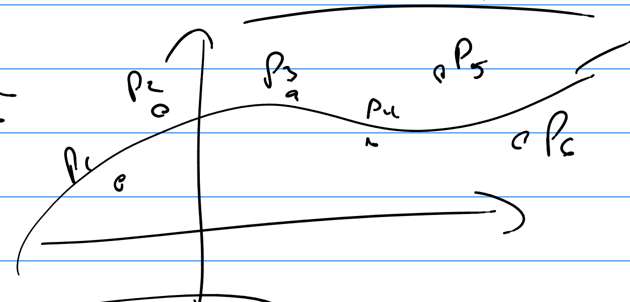
$N(A) = \text{span}(?)$

$\rightarrow R(A) = \text{span}(?)$

$N(A^T) = \text{span}(?)$

5.3 least-squares soln's (data fit problem)

Set up the problem



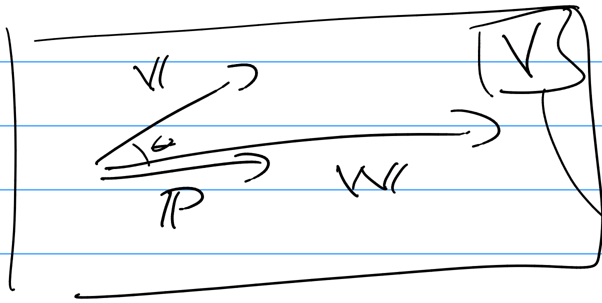
fit by some polynomial.

$\rightarrow \boxed{Ax = b}$ over det. prob.

$A^T A x = A^T b$

$\hat{x} = (A^T A)^{-1}(A^T b)$

5.4 (2 probs) (like 5.1 prob + projection)

① 

$p = ?$

P_3 space

② $\mathbb{R}^{m\times n}$ space find θ between two matrices

ex (non-weighted inner product)

$\theta = \cos^{-1}(\quad)$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

5.5 (2 probs) orthonormal

① verify matrices are orthonormal.

② use coord. of orthonormal basis to do an integral.

5.6 given matrix find $\boxed{q_1, q_2, \ldots}$ using gram-schmidt

3x3