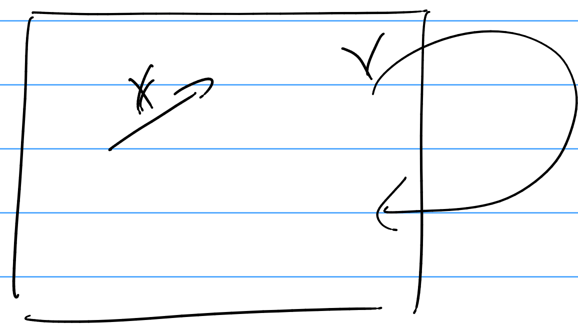
Math 511

Eigen Values / Eigen Vectors / Eigen Spaces

Concept: L is a linear operator $L: \mathbb{R}^n \to \mathbb{R}^n$



Matrix A is L's matrix representation

$$L(x) = Ax$$

(ex) Markov Process $x_0,\ x_1 = Ax_0,\ x_2 = Ax_1 = A^2x_0$

$$x_3 = Ax_2 = A^3x_0$$

$$\vdots$$

$$x_k = Ax_{k-1} = A^k x_0$$

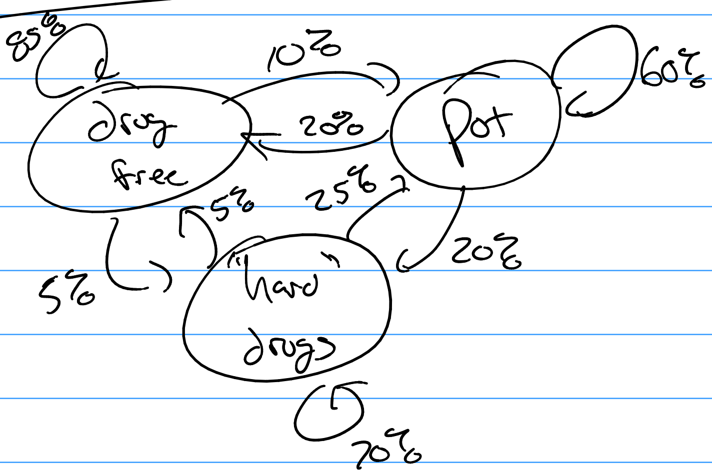
but doing prob's like this we see..

$$\underline{Ax} = \underline{\lambda x} \quad \text{for special } \lambda, x$$

Can I find the λ's (Eigen Values)

and for each λ find its x (Eigen vector associated with λ)

(ex)



$$x_0 = \begin{bmatrix} .70 \\ .25 \\ .05 \end{bmatrix} \begin{matrix} \leftarrow \text{drug free} \\ \leftarrow \text{pot} \\ \leftarrow \text{"hard" drugs} \end{matrix}$$

$$x_1 = Ax_0$$

$$x_2 = Ax_1$$

$$\vdots$$

$$A = \begin{bmatrix} .85 & .20 & .05 \\ .10 & .60 & .25 \\ .05 & .20 & .70 \end{bmatrix}$$

$$\lambda^3 \quad \lambda^2 \quad \lambda \quad = 0$$

Find out λ — need to learn this part

$$A\,b_1 = 1 \cdot b_1 \qquad b_1 = \begin{bmatrix} -.768 \\ -.466 \\ -.439 \end{bmatrix}$$

$$\boxed{A\,b_2 = .743 \cdot b_2} \qquad b_2 = \begin{bmatrix} -.809 \\ .309 \\ .5 \end{bmatrix}$$

$$A\,b_3 = .407 \cdot b_3 \qquad b_3 = \begin{bmatrix} .309 \\ .809 \\ .5 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} .7 \\ .25 \\ .05 \end{bmatrix}_E$$

$$[X_0]_B = B^{-1}\begin{bmatrix} .7 \\ .25 \\ .05 \end{bmatrix}_B = \begin{bmatrix} -.60 \\ -.33 \\ -.09 \end{bmatrix}_B = \boxed{-.60\,b_1 - .33\,b_2 - .09\,b_3}$$

why?: $X_k = A^k X_0 = A^k(\ [\ \ ]\ )$

$$X_k = -.60\,b_1 - .33\,\cancel{(.743)^k}\,b_2 - .09\,\cancel{(.407)^k}\,b_3 \qquad (.743)^k \to 0,\ (.407)^k \to 0$$

$$X_k \Rightarrow -.60\,b_1 = \begin{bmatrix} .46 \\ .28 \\ .26 \end{bmatrix}$$

$k \to \infty$

← free
← pot
← hard drugs

Finding Eigen Values / Eigen Vectors

goal $\quad A X = \lambda X$

$$A X - \lambda X = 0$$

$$(A)X - (\lambda I)X = 0$$

$$(A - \lambda I)X = 0 \qquad \text{homogeneous system}$$

(1) $AX = \lambda X$
is $\quad (A - \lambda I)X = 0$

(2) want to find λ for non-trivial soln

(3) $(A - \lambda I)$ must be singular

(4) $\det(A - \lambda I) = 0$

So to find eigen values (λ) we solve $\det(A-\lambda I)=0$ for λ's.

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

Step 1 $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ -1 & 1-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{bmatrix}$

Step 2 solve $\det\begin{pmatrix} 1-\lambda & 2 & 3 \\ -1 & 1-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{pmatrix} = 0$

polynomial of λ with degree of 3.

solve $\lambda_1 = ?,\ \lambda_2 = ?,\ \lambda_3 = ?$

Step 3 for each λ_i (eigenvalue) to find its v_i (eigen vector)

Solve: $(A - \lambda_i I)X = 0$ (Null space)

vectors of Null space are the eigen vectors ex $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Step 1 $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

Step 2 $\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$

$$(1-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\lambda = 1 \quad \lambda = 2 \quad \lambda = 1$$

Step 3 for each $\lambda_1 = 1$ $\lambda_2 = 2$

a) $\lambda_1 = 1$ Solve $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = -\beta$

free free

$x_1 = \alpha$ $x_3 = \beta$

$$x = \begin{bmatrix} \alpha \\ -\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1$ its eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

or eigen space is $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$

b) $\lambda_2 = 2$ etc

Properties of eigen values / vectors

① $\lambda_1 \lambda_2 \cdots \lambda_n = \det(A)$

② $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \underline{a_{11} + a_{22} + \cdots + a_{nn}}$ = trace of A
$= tr(A)$

③ if $\lambda = a+bi$ is an eigen value
$\underline{\underline{\text{then}}}$ $\lambda = a-bi$ is $\underline{\underline{\text{also}}}$ an eigen value

denote complex conj, $\overline{a+bi} = a-bi$

$\underline{\underline{\text{then:}}}$ λ_1 has v_1 eigen vector
$\overline{\lambda_1}$ has $\overline{v_1}$ eigen vector

$\lambda = 2+i$ $v = \begin{bmatrix} 2+0i \\ 1-3i \end{bmatrix} \xrightarrow{\text{know}}$ $\lambda = 2-i$ $v = \begin{bmatrix} 2-0i \\ 1+3i \end{bmatrix}$

Note: $\underline{\underline{v = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}}}$ $\underline{\underline{v = \begin{bmatrix} 2i \\ 3+i \end{bmatrix}}}$

Same eigen vectors!
