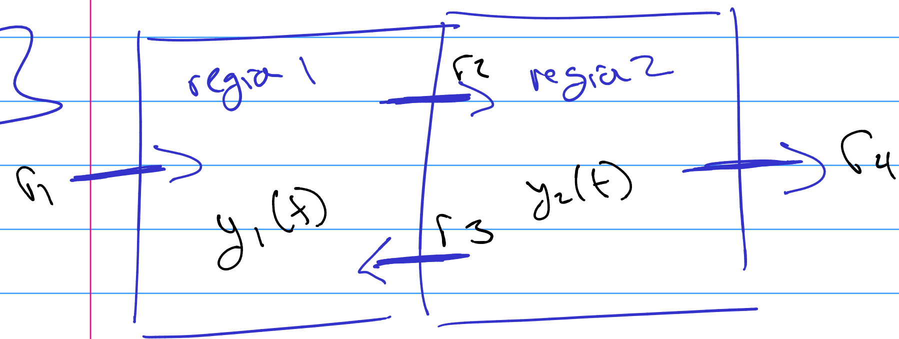
Math 511

ex



rates r_1, r_2, r_3, r_4

amounts $y_1(t)$ $y_2(t)$

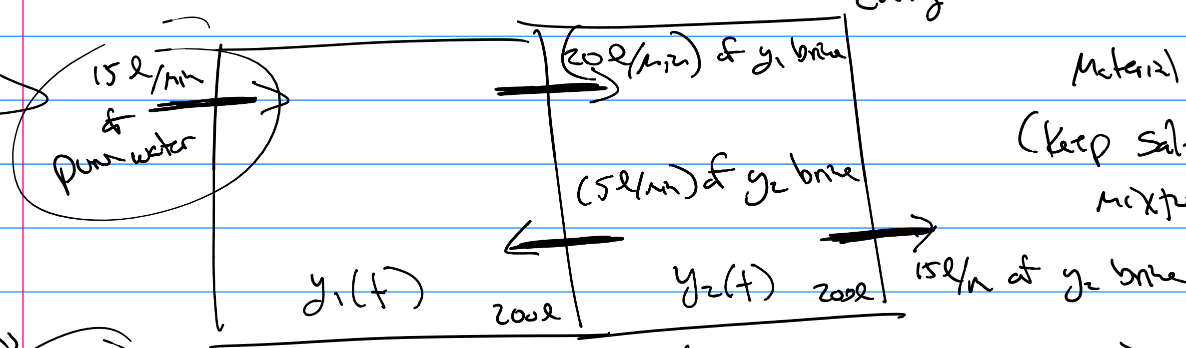
1^{st} order system of linear diff. eqn's

$$\frac{dy_1}{dt} = (\;)y_1 + (\;)y_2$$

based rates

rate of change = (going into) − (going out of)

ex



Material = Salt (grams)

(Keep salt in a brine mixture)

$\frac{\Delta \text{salt}}{\Delta \text{time}}$

$$\frac{dy_1}{dt} = (\text{in}) - (\text{out}) = \left(15\frac{l}{min}\cdot\frac{0g}{l} + 5\frac{l}{min}\cdot\frac{y_2\, g}{200\, l}\right) - \left(20\frac{l}{min}\cdot\frac{y_1\, g}{200\, l}\right)$$

$$\frac{dy_1}{dt} = \left(\frac{1}{40}y_2 - \frac{1}{10}y_1\right) g/min$$

$$\frac{dy_2}{dt} = \left(20\frac{l}{min}\frac{y_1\, g}{200\, l}\right) - \left(20\frac{l}{min}\cdot\frac{y_2\, g}{200\, l}\right) = \left(\frac{1}{10}y_1 - \frac{1}{10}y_2\right) g/min$$

So we have..

$$\boxed{\begin{aligned} y_1' &= -\frac{1}{10}y_1 + \frac{1}{40}y_2 \\ y_2' &= \frac{1}{10}y_1 - \frac{1}{10}y_2 \end{aligned}}$$

1^{st} order linear diff. eqn

Solve?: $y_1(t) = \boxed{?}$ $y_2(t) = \boxed{?}$

$\boxed{\text{Guess / check}}$ try a similar yet easier problem.

ex 1 eqn 1 unknown: $y' = ay$

guess $y = e^{at} \to \boxed{y'} = ae^{at} = \boxed{ay}$

So general soln is $y = Ce^{at}$

back to our prob: $y_1' = -\frac{1}{10}y_1 + \frac{1}{40}y_2$

$y_2' = \frac{1}{10}y_1 - \frac{1}{10}y_2$

let $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} -1/10 & \frac{1}{40} \\ 1/10 & -1/10 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y' = AY$$

Guess: each $y_i = e^{(\text{const})t}$ or λ eigenvalue with X as its eigenvector

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = e^{\lambda t} X = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix}$$

check: a) $Y' = \lambda \boxed{e^{\lambda t} X} = \boxed{\lambda Y}$

b) $AY = A(e^{\lambda t} X) = e^{\lambda t}(AX) = e^{\lambda t}(\lambda X) = \lambda(e^{\lambda t} X)$
$= \boxed{\lambda Y}$

So $Y' = AY$!

So
$$y_1' = a_{11}y_1 + \dots + a_{1n}y_n$$
$$y_2' = a_{21}y_1 + \dots + a_{2n}y_n \rightarrow Y' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & \ddots & \\ a_{n1} & & \dots & a_{nn} \end{bmatrix} Y$$
$$\vdots$$
$$y_n' = a_{n1}y_1 \dots + a_{nn}y_n$$

$\rightarrow$ we can find each λ_i eigen value with its X_i eigen vector

Soln $\quad Y = e^{\lambda t} X$ for each λ_i

general Soln $\quad Y = C_1 e^{\lambda_1 t} X_1 + C_2 e^{\lambda_2 t} X_2 + \dots + C_n e^{\lambda_n t} X_n$

for constants $C_1, C_2, \dots, C_n$

Initial value problem $\quad$ given @ $t=0$ $\quad Y(0) = Y_0$ (tell amounts @ start)

$\rightarrow$ use this to find specific $C_1, C_2, \dots C_n$

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back to example: $\quad y_1' = -\frac{1}{10} y_1 + \frac{1}{40} y_2$ $\quad$ so $A = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}$

$\quad y_2' = \frac{1}{10} y_1 - \frac{1}{10} y_2$

Soln: $\quad C_1 e^{\lambda_1 t} X_1 + C_2 e^{\lambda_2 t} X_2$ for eigen's λ_1, X_1
λ_2, X_2

Eigen Values / Vectors $\quad \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}$

(1) $A - \lambda I = \begin{bmatrix} -1/10 - \lambda & 1/40 \\ 1/10 & -\frac{1}{10} - \lambda \end{bmatrix}$

(2) $\det(A - \lambda I) = 0$ $\quad$ solve for λ

$$\left(-\frac{1}{10} - \lambda\right)^2 - \frac{1}{400} = 0 \rightarrow \left(-\frac{1}{10} - \lambda\right)^2 = \frac{1}{400}$$

$$\frac{1}{100} + \frac{\lambda}{5} + \lambda^2 - \frac{1}{400} = 0$$

$$4 + 80\lambda + 400\lambda^2 - 1 = 0$$

$$400\lambda^2 + 80\lambda + 3 = 0$$

$$\lambda = \frac{-80 \pm \sqrt{(80)^2 - 4(400)(3)}}{2(400)}$$

$\vdots$

$$\lambda_1 = -\frac{1}{20} \qquad \lambda_2 = -3/20$$

$$\left(-\frac{1}{10} - \lambda\right) = \pm\sqrt{\frac{1}{400}}$$

$$-\frac{1}{10} - \lambda = \pm \frac{1}{20}$$

$$\lambda = -\frac{1}{10} \pm \frac{1}{20}$$

$$\lambda_1 = -\frac{1}{10} + \frac{1}{20} = -\frac{1}{20}$$

$$\lambda_2 = -\frac{1}{10} - \frac{1}{20} = -\frac{3}{20}$$

③ for each λ_i, find X_i solve $\left[A - \lambda_i I \,\middle|\, 0\right]$

a) $\lambda_1 = -\frac{1}{20}$

$$\left[\begin{array}{cc|c} -\frac{1}{10} + 1/20 & 1/40 & 0 \\ 1/10 & -\frac{1}{10} + 1/20 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -\frac{1}{20} & \frac{1}{40} & 0 \\ \frac{1}{10} & -\frac{1}{20} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to X = \begin{bmatrix} \frac{1}{2}\alpha \\ \alpha \end{bmatrix}$$

free!

$X_2 = \alpha$
$X_1 = \frac{1}{2}\alpha$

$$\lambda_1 = -1/20 \quad X_1 = \cancel{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$